# VIANA MAPS DRIVEN BY BENEDICKS-CARLESON MAPS

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## DECLARATION

I hereby declare that the thesis is my original work and it has been written by me in its entirety.

I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

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01 July 2014

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# Contents

# Summary

In the research area of dynamical systems with hyperbolic behaviors, Viana maps refer to a class of dynamical systems named after Marcelo Viana, which are skew-products of quadratic maps driven by expanding maps. In this thesis, we consider a family of Viana maps that are constructed by coupling two quadratic maps. We are devoted to studying the measurable dynamics of this family, especially the abundance of non-uniform hyperbolicity in this family. We prove that, for any polynomial coupling function of odd degree, when the parameter pair of the two factor quadratic maps is chosen from a two-dimensional positive measure set, the associated Viana map has two positive Lyapunov exponents and admit finitely many ergodic absolutely continuous invariant probability measures. The main ingredient in the thesis is concentrated in Chapter 3, where we make use of complex analytic techniques to deduce non-flatness for iteration image of horizontal curves, by taking advantage of Benedicks-Carleson condition on the driven map and nice analytical properties of the coupling function.

Chapter 1

# Introduction

## 1.1 Background

### 1.1.1 Viana map driven by a circle expanding map

Hyperbolicity is a significant feature for various dynamical systems and it plays a central role in mathematical understanding of their statistical behaviors. The notion of uniform hyperbolicity for general dynamical systems was proposed by Smale in 1960s, see [Sma67]. Later on in 1970s, uniform hyperbolic dynamical systems were extensively studied in the statistical viewpoint by Sinai, Ruelle, Bowen and other authors, see [Sin72, Rue76, BR75, Bow75], etc. These works establish the concept of physical/SRB measure and help people to well understand generic dynamical systems exhibiting uniform hyperbolicity.

Non-uniform hyperbolicity, primarily characterized by non-zero Lyapunov exponents, is a more common phenomenon beyond uniform hyperbolicity and it is possessed by much broader classes of dynamical systems. On the one hand, in one dimensional case, due to the pioneering work of Jakobson in [Jak81] and subsequent works [CE83, BC85], etc, the non-uniform hyperbolicity is well understood and known to be abundant. On the other hand, in multi-dimensional case, although the theoretical study of general non-uniform hyperbolic dynamical systems was initiated by Pesin in [Pes76] as early as in 1970s, which is based on Oseledets' work in [Ose68], it is still far from being complete till now. Even in the non-uniformly expanding case, i.e. when the Lyapunov exponent along each direction is positive, the picture is quite unclear.

In constructing non-uniformly expanding maps in dimension greater than one, a natural idea is trying to couple uniformly/non-uniformly expanding one-dimensional maps with non-uniformly expanding one-dimensional maps. Viana successfully applied this idea by introducing skew-product of a quadratic map driven by a circle expanding map as follows in [Via97].

$$F : S^1 \times \mathbb{R} \circlearrowleft, \quad (\theta, y) \mapsto (g(\theta), Q_b(y) + \alpha\varphi(\theta)). \tag{1.1}$$

Here $S^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle, $g(\theta) = d\theta$ for some integer $d \geq 2$, and $Q_b(x) = b - x^2$ is a Misiurewicz-Thurston quadratic map for some $b \in (1, 2)$, and $\varphi$ is some $C^3$ function with nice analytic properties(a typical example is $\varphi(\theta) = \sin(2\pi\theta)$). Then he proved that under certain restrictions on $d$ and $\varphi$, $F$ has two positive Lyapunov exponents almost everywhere when $\alpha > 0$ is small.

Due to Viana's pioneering work, nowadays a skew-product system of similar form to (1.1) is called a Viana map. For such a map, we will call its first coordinate the horizontal direction and its second coordinate the vertical direction. The factor map along the horizontal direction will be called base/driven dynamics, and the other factor will be called fibre/skew-product part. The function $\varphi$ appearing in the perturbation term is called the coupling function.

In improving Viana's result, several subsequent works were done in the last decade, either focusing on improving the lower bound of $d$ or trying to relax the restriction on $\varphi$, and all of these works lead to two positive Lyapunov exponents result. Let us summarize these works together as follows for comparison.

- In [Via97], Viana assumed that $d \geq 16$ and $\varphi(\theta) = \sin(2\pi\theta)$, and his argument still functions well when $\sin(2\pi\theta)$ is replaced by general $C^3$ function without degenerate critical point. Moreover, the same statement holds for small $C^3$ perturbation of $F$.
- In [BST03], Buzzi, Sester and Tsujii weakened the lower bound of $d$ to the natural bound $d \geq 2$ for the coupling function $\varphi(\theta) = \sin(2\pi\theta)$. Moreover, the same statement holds for small $C^\infty$ perturbation of $F$.
- In [Sch08], Schnellmann considered non-integer case of $d > 1$ for which the vertical expansion is dominated by the horizontal expansion, and obtained the same result on two positive Lyapunov exponents.
- In [HS13], Huang and Shen generalized the result in [BST03] to arbitrary non-constant real analytic coupling function $\varphi$, under the same assumption $d \geq 2$ being an integer.

### 1.1.2 Review of Viana and other's argument

As in other previous works on Viana maps reviewed before, in this thesis we will follow the basic framework of Viana's original argument in [Via97]. Therefore, let us briefly review some of its key points. In Viana's original construction of (1.1) and in subsequent works on Viana maps, in addition to the skew-product form of $F$ and the fact that both factor maps of $F$ are showing uniformly/non-uniformly expanding behavior, there are two most important features as follows that should be highlighted.

(PH) (Partial hyperbolicity) The expansion in the vertical direction is dominated by the expansion in the horizontal direction in the sense that the lower bound of the expansion speed in the horizontal direction is faster than the upper bound of the expansion speed in the vertical direction.

(NF) (Non-flat coupling) The coupling function $\varphi$ is far from being flat, in the sense that under (long time) iteration of $F$, the image of a horizontal curve(the so called admissible curve), say $Y$, becomes sufficiently non-flat, i.e. for every $\theta \in S^1$, $|Y^{(l)}(\theta)| \geq \delta\alpha$ (here $Y^{(l)}$ denotes the $l$-th derivative of $Y$) for some relatively small $l \in \mathbb{N}$ and some relatively large $\delta > 0$.

These two features provide a quantitative control of the proportion in the graph of an admissible curve passing through the $\alpha\epsilon$-neighborhood of the critical line $S^1 \times \{0\}$ for $\epsilon > 0$ small, which plays a central role in deducing the slow recurrence condition in the vertical direction.

Besides (PH) and (NF), there are something in common for all the works that is worth mentioning. Firstly, thanks to the skew-product structure of $F$ and the non-uniformly expanding behavior of $Q_b$, the positiveness of the vertical Lyapunov exponent can be implied by the slow recurrence condition in the vertical direction. Secondly, by Fubini's theorem, verifying the slow recurrence condition in the vertical direction can be reduced to analyzing the iteration of each single horizontal curve.

Since we will also use many ideas in previous works beyond Viana's original setting, let us make comparison between the main difference of different settings. In [Via97], mainly because $d$ is much larger than the supremum of $|Q_b'|$, the non-flatness of an admissible curve $Y$ can be detected by only considering at most second order derivative of $Y$. In [BST03], since the expansion in the base dynamics becomes much weaker, they have to take higher order derivatives of $Y$ into consideration. Taking advantaging of their specific choice of $\varphi$, this can be handled by considering at most $l$-th derivative of $Y$ for some definite $l \in \mathbb{N}$. However, their approach highly relies on the particular properties of $\varphi(\theta) = \sin(2\pi\theta)$. In [HS13], to deal with general analytic coupling function $\varphi$, they introduced complex analysis argument and also showed that up to taking $l$-th derivative for some definite $l \in \mathbb{N}$, the non-flatness can be revealed. However, for general $\varphi$, some resonance phenomenon between $d$ and the period of $\varphi$ occurs, which prevents them from using the argument in [Via97, BST03] to control the recurrence to $\alpha^{1-\eta}$-neighborhood of $S^1 \times \{0\}$ for $\eta > 0$ small. They used a different approach to avoid this problem, and the cost is that the lower bound of vertical Lyapunov exponent becomes dependent on $\alpha$.

### 1.1.3 About a.c.i.p.'s of Viana maps

The existence of two positive Lyapunov exponents for Viana maps is a strong hint for existence of absolutely continuous invariant probability measure(a.c.i.p. for short) and other statistical properties beyond. As a first step toward this direction, in [Alv00], Alves proved that $F$ in (1.1) admits finitely many ergodic a.c.i.p.'s under the setting of [Via97] by studying so called "hyperbolic times". Then the uniqueness of a.c.i.p. was proved in [AV02] under the same setting.

In [ABV00], the hyperbolic-time technique in [Alv00] was abstracted by Alves, Bonatti and Viana to prove existence of a.c.i.p.'s for general non-uniformly expanding maps which satisfy an assumption on positive Lyapunov exponent and an additional assumption on slow recurrence of orbits to the critical/singular set. One may also see [Alv06] for a comprehensive presentation of the work on non-uniformly expanding maps in [Alv00] and for discussion of further statistical properties of Viana maps beyond existence of a.c.i.p.'s.

More recently, in [Sol13], Solano proved that for a two-dimensional partially hyperbolic skew-product map driven by a circle-expanding map, the slow recurrence condition in [ABV00] is redundant for deducing existence of a.c.i.p.'s. As a generalization of this result, in [AS11], Araujo and Solano showed that for a two-dimensional skew-product system $F$ with general base dynamics that admits an a.c.i.p., to obtain the existence of a.c.i.p.'s for $F$, it is sufficient to assume that the vertical Lyapunov exponent is almost where positive.

Beyond the works reviewed above, it is worth noticing a related work of Tsujii on partially hyperbolic surface endomorphisms. In [Tsu05], Tsujii showed that for a generic sufficiently smooth partially hyperbolic surface endomorphism with one uniformly expanding direction, it admits finitely many ergodic physical measures, and the union of their basins has full Lebesgue measure. One of the key steps in Tsujii's argument was motivated by the idea of the studying non-flatness of admissible curves in Viana [Via97].

### 1.1.4 Viana map driven by a non-uniformly expanding map

It should be noted that in all the works reviewed in § 1.1.1, the base dynamics are assumed to be uniformly expanding. As a first step in relaxing the expansion of the horizontal direction to non-uniform case, in [Sch09], Schnellmann considered to use a Misiuremicz-Thurston polynomial instead of a circle expanding map as the base dynamics. He studied the following skew-product system:

$$F : [a - a^2, a] \times \mathbb{R} \circlearrowleft \quad , \quad (x, y) \mapsto (g_a(x), Q_b(y) + \alpha\varphi(x)), \tag{1.2}$$

where $1 < a \leq 2$, $1 < b < 2$, $g_a = Q_a^m$, $Q_a$ and $Q_b$ are Misiurewicz-Thurston, and $m$ is a large positive integer so that the partial hyperbolicity of $F$ can be easily verified. For certain coupling function $\varphi$, he also proved that $F$ in (1.2) has two positive Lyapunov exponents almost everywhere, for $\alpha > 0$ small enough. In proving his result, Schnellmann made the basic observation that when $Q_a$ is Misiurewicz-Thurston, its unique a.c.i.p. induces an real analytic change of coordinates except for a finite number of mild singularities, which conjugates $Q_a$ to a uniformly expanding map. This helps him to obtain partial hyperbolicity of $F$ and to prove his result.

However, the disadvantage in Schnellmann's construction is that the coupling function $\varphi$ has singularities and its choice depends on $Q_a$. In [GS14], Shen and the author proved that for $F$ of the same form as given in (1.2), for $\varphi$ being a non-constant polynomial of odd degree, when $\alpha > 0$ is sufficiently small, $F$ admits two positive Lyapunov exponents. In their argument, in obtaining non-flatness of admissible curves, they adopted the basic idea of [HS13], while in dealing with vertical recurrence in $\alpha^{1-\eta}$-neighborhood of $[a - a^2, a] \times \{0\}$, they followed [Via97, BST03].

Although the expansion of the base dynamics in [Sch09, GS14] looks in a non-uniform way, it is not far from being uniform, as shown in their proofs. Besides, there are only countably many Misiurewicz-Thurston parameters in the quadratic family. Therefore, a natural question is to consider more typical one-dimensional maps with an essential non-uniform way of expansion as base dynamics. Thanks to [Jak81, CE83, BC85] and subsequent works, we know that Collet-Eckmann quadratic maps are typical non-uniformly expanding maps in the quadratic family and they can serve as good candidates for base dynamics for Viana maps, which inspires the basic motivation of this thesis.

## 1.2 Statement of results

Before stating our result, let us recall the concepts of Lyapunov exponent and absolutely continuous invariant probability measure. Let $(M, g)$ be a Riemannian manifold (either with or without boundary). Let $\|\cdot\|$ denote the norm induced by $g$, and let us call the volume measure on $M$ induced by $g$ the Lebesgue measure. Let $f : M \circlearrowleft$ be a differentiable map and let $\mathrm{D}f$ denote its tangent map.

Given $p \in M$ and $v \in T_pM \setminus \{0\}$, define the **Lyapunov exponent** of $f$ at $p$ along $v$, denoted by $\chi(p, v)$ as follows, if the limit in the definition does exist.

$$\chi(p, v) := \lim_{n\to\infty} \frac{1}{n} \log \frac{\|\mathrm{D}f^n(v)\|}{\|v\|}.$$

Given $1 \leq k \leq \dim M$, we say that $f$ has $k$ Lyapunov exponent at Lebesgue almost every point, if there exists an $f$-invariant sub-bundle $S$ of the tangent bundle $TM$ with dimension $k$, such that $\chi(p, v)$ exists for Lebesgue almost every point $p \in M$ and for every $v \in (S \cap T_p M) \setminus \{0\}$.

Let $\mu$ be a Borel probability measure on $M$. We say that $\mu$ is **$f$-invariant**, if for every Borel set $E \subset M$, $\mu(E) = \mu(f^{-1}(E))$. We say that $\mu$ is an **absolutely continuous invariant probability measure**(a.c.i.p. for short) with respect to $f$, if $\mu$ is both $f$-invariant and absolutely continuous with respect to the Lebesgue measure on $M$.

Let us parameterize the family of complex quadratic polynomials whose unique critical point is located at the origin in the following way: $Q_c(z) := c - z^2$, $c \in \mathbb{C}$. When $c \in \mathbb{R}$, $Q_c$ can also be considered as a map of the real line to itself. In particular, when $c \in (1, 2]$, $Q_c$ induces an S-unimodal map( here 'S' represents negative Schwarz derivative; see, for example, [BL91] for more details) as follows:

$$Q_c : [-\beta_c, \beta_c] \circlearrowleft , \quad \text{where} \quad \beta_c := \frac{1 + \sqrt{1 + 4c}}{2}.$$

Here the invariant interval of $Q_c$ is so chosen as to follow the convention that its boundary consists of its unique orientation preserving (and hyperbolic repelling) fixed point of $Q_c$, namely $-\beta_c$, and its another pre-image, namely $\beta_c$.

Now we can state the main result in this thesis. Let us consider the skew-product of two quadratic maps defined as follows:

$$F = F_{a,b,\alpha} : [-\beta_a, \beta_a] \times \mathbb{R} \circlearrowleft , \quad (x, y) \mapsto (Q_a(x), Q_b(y) + \alpha\varphi(x)), \tag{1.3}$$

where $(a, b) \in (1, 2] \times (1, 2)$, $\alpha > 0$ is small and $\varphi$ is a non-constant polynomial.

**Main Theorem.** *Suppose that the polynomial $\varphi$ appearing in* (1.3) *is of odd degree. Let* $(a, b)$ *be taken from the parameter set $\mathscr{P}$ characterized in* (2.15)*, which is a Borel set of positive two-dimensional Lebesgue measure. Then there exists $\alpha_{a,b} > 0$, such that when $0 < \alpha < \alpha_{a,b}$, the map $F_{a,b,\alpha}$ defined in* (1.3) *has two Lyapunov exponents of positive* *lower bound independent of $\alpha$ at Lebesgue almost every point, and it admits finitely many ergodic a.c.i.p.'s. Moreover, the number of ergodic a.c.i.p.'s of $F_{a,b,\alpha}$ does not exceed that of $Q_a \times Q_b$.*

*Remark.*

- By checking the proof of the theorem carefully, we can see that $\mathscr{P}$ can be decomposed into a countable union of product types: $\mathscr{P} = \bigcup_{n=1}^{\infty} \mathscr{A}_n \times \mathscr{B}_n$, where $\bigcup_{n=1}^{\infty} \mathscr{A}_n = \mathscr{C}_{BC}$ is defined in 2.1.1; moreover, for every $n \in \mathbb{N}$, there exists $\alpha_n > 0$, such that there exists a uniform positive lower bound for the two Lyapunov exponents of $F_{a,b,\alpha}$ when $(a, b, \alpha) \in \mathscr{A}_n \times \mathscr{B}_n \times (0, \alpha_n)$.

- If the parameter $b$ is not Misiurewicz-Thurston, then $F_{a,b,\alpha}$ always has exactly the same number of ergodic components of a.c.i.p.'s as $F_{a,b,0} = Q_a \times Q_b$. See Corollary 5.2.2 for details. In particular, there exists a full measure subset $\mathscr{P}_0$ of $\mathscr{P}$, such that for $(a, b) \in \mathscr{P}_0$, $F_{a,b,\alpha}$ has the same number of ergodic a.c.i.p.'s as $F_{a,b,0}$ when $\alpha > 0$ is small. Furthermore, there exists a positive measure subset of $\mathscr{P}_0$, such that for $(a, b)$ is this set, $F_{a,b,\alpha}$ has a unique a.c.i.p. .

- The assumption that $\varphi$ is of odd degree is inherited from the same assumption in [GS14] for the Misiurewicz-Thurston case of $Q_a$. This assumption cannot be fully gotten rid of when proving (3.3) in Lemma 3.1.3(see remark under the lemma), which plays a central role in deducing (3.13) and Proposition 4.2.1 accordingly.

## 1.3 Outline

### 1.3.1 Brief review of proof of the main theorem

As mentioned before, our argument follows the basic framework of [Via97] among others. Here let us only highlight how we deal with the two features (PH) and (NF) in our work.

To guarantee partial hyperbolicity of $F$, compared with [Sch09, GS14], an improvement in this thesis is that we can simply use $Q_a$ as the base dynamics rather than its high iteration. This is achieved by choosing $b$ as a sufficiently many times renormalizable map and restricting it onto an small invariant set. Thanks to the abundance of non-uniformly expanding parameters in the quadratic family, for every Collet-Eckmann $Q_a$ the associated parameter $b$ forms a positive measure set, see Lemma 2.1.4. The cost of our improved setting is that the domination relation between $Q_a$ and $Q_b$ can not be directly reflected in comparing their derivatives in a definite long time iteration, and therefore, the condition on positive Lyapunov exponent in [ABV00] becomes tricky to verify. As a result, we use the result [AS11] to conclude that $F$ admits only finitely many a.c.i.p.'s from Proposition 5.1.1 instead, and mimic the argument in [ABV00] to discuss the number of ergodic a.c.i.p.'s, see Proposition 5.2.1.

A more important difference between our setting and the previous ones is that now the base dynamics $Q_a$ is not uniformly expanding, even under smooth coordinate change with mild singularities. As a result, given any $l_0 \in \mathbb{N}$, for a curve $Y$ arising from (long time) $F$-iteration of a horizonal curve, it is hard to detect its non-flatness by only considering $l$-th derivative of $Y$ for $1 \le l \le l_0$. However, here is a simple but key observation to overcome this difficulty: if we allow to consider $Y^{(l)}$ for arbitrary $l \in \mathbb{N}$, then no matter how small the domain of $Y$ is, we will ultimately observe certain non-flatness that is sufficient for our usage. This basic idea is reflected in the statement of Proposition 3.4.4. Let us make a little more explanation to show how this idea is realized. Thanks to the polynomial form of $Q_a$, $Q_b$ and $\varphi$, $Y$ can always extend to a holomorphic function on a neighborhood $U \subset \mathbb{C}$ of its domain $I \subset \mathbb{R}$. Then due to a compactness argument for holomorphic functions, see 3.4.2, roughly speaking, the non-flatness of $Y$ can be obtained by showing that the ratio between the oscillation of $Y$ on $U$(denoted by $O_U$) and the oscillation of $Y$ on $I$(denoted by $O_I$) is not to large. The lower bound of $O_I$ is obtained in more or less the same way as [HS13, GS14], see Lemma 3.2.2; to control the upper bound of $O_U$, technically we assume that the recurrence of critical orbit of $Q_a$ is at most stretched exponentially fast, see Lemma 3.3.2 and Lemma 3.3.3. Although the estimate in Proposition 3.4.4 seems not as strong as similar estimates in previous works, it does not weaken the tail estimate of slow recurrence in the vertical direction much, see Proposition 4.3.1.

### 1.3.2 Organization of the thesis

Following the introduction chapter, the main body of thesis consists of four chapters and they are organized as follows.

In Chapter 2, as indicated in its title, we collect useful materials for future use. In § 2.1, we begin with summarizing part of celebrated results about abundance of non-uniform hyperbolicity in real quadratic family, and then have some discussion on renormalization and Collet-Eckmann condition for the real quadratic family. In particular, for the implications of the Collet-Eckmann condition, we mainly use the "semi-uniformly expanding" property for building vertical expansion and the "exponentially shrinking" for studying admissible curves and for discussing the a.c.i.p.'s of $F$. Once the review of the quadratic family is done, we are ready to give the description of our parameter set $\mathscr{P}$ in (2.15). In section 2.2, we establish basic estimates on iteration of $F$ for the main body of our argument, both from the random perturbation viewpoint in the real domain and from the partial hyperbolicity of $F$ in the complex domain. These elementary estimates will be frequently used through most part of the argument. In § 2.3, following [Via97], we build vertical expansion for orbits of $F$ outside $\sqrt{\alpha}$-neighborhood of $[-\beta_a, \beta_a] \times \{0\}$.

In Chapter 3, following all the previous works on Viana maps, we introduce the concept of admissible curves, and study their analytic properties. The main result in this chapter is Proposition 3.4.4, which gives quantitative description control of iteration of an admissible curve passing through $\alpha\epsilon$-neighborhood of the critical line. As mentioned before, this proposition will play a central role in proving Proposition 4.3.1. In § 3.1, following the approach in [HS13] and especially [GS14], we introduce a family of analytic functions $\mathcal{T}_r$ and prove that it is non-degenerate. Then in § 3.2, given an $r$-admissible curve $Y$, we choose appropriate $T \in \mathcal{T}_r$ to approximate the derivative of $Y$ by $\alpha T$, which gives deserved lower bound of oscillation of $Y$ in Corollary 3.2.2. In § 3.3, with the help of the stretched exponential recurrence assumption on the critical orbit of $Q_a$, we obtain desired upper bound of oscillation of $Y$ in Lemma 3.3.3. Combing this with the lower bound we finally prove Proposition 4.3.1 in § 3.4.

In Chapter 4, we are devoted to establishing the slow recurrence condition in the vertical direction, and as a byproduct, a stretched exponential tail is obtained, see Proposition 4.3.1. In § 4.1, we introduce an induced Markov of $Q_a$ with bounded distortion and exponential small tail for convenience to apply large deviation estimates in the rest of this chapter. In § 4.2, we adopt the approach in [Via97, BST03, GS14] to deal with vertical recurrence to the $\alpha^{1-\eta}$-neighborhood of the critical line, where as in [GS14], we have to assume that $\varphi$ is of odd degree. In § 4.3, we combine control on vertical recurrence in both Proposition 3.4.4 and Proposition 4.2.1, and then use a large deviation argument as in [Via97] to complete the proof of Proposition 4.3.1.

Finally, in Chapter 5, we complete the proof of the Main Theorem. In § 5.1, we prove that $F$ has two positive Lyapunov exponents and admits finitely many ergodic a.c.i.p.'s. In § 5.2, we use the approach in [ABV00] to prove the statement on number of ergodic a.c.i.p.'s of $F$ in the Main Theorem.

## 1.4 Notations

Given $x \in \mathbb{R}$ and $r > 0$, denote $\mathbb{I}(x, r) := (x - r, x + r)$ and $\mathbf{I}(x, r) := [x - r, x + r]$; moreover, denote $\mathbb{I}_r := \mathbb{I}(0, r)$ and $\mathbf{I}_r := \mathbf{I}(0, r)$ for short. Similarly, given $z \in \mathbb{C}$ and $r > 0$, denote $\mathbb{D}(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$ and $\mathbf{D}(z, r) = \{w \in \mathbb{C} : |w - z| \leq r\}$; moreover, denote

$\mathbb{D}_r := \mathbb{D}(0, r)$ and $\mathbf{D}_r := \mathbf{D}(0, r)$ for short.

When there is no ambiguity caused by the parameter $c \in \mathbb{C}$ of $Q_c$, let us denote the connected component of $Q_c^{-n}(\mathbb{D}(Q_c^n(z), r))$ containing $z$ by $\mathrm{Comp}(z, n, r)$ for short, where $Q_c$ is considered as a holomorphic map on $\mathbb{C}$, $z \in \mathbb{C}$, $n \in \mathbb{N}$ and $r > 0$. Note that $\mathrm{Comp}(z, n, r)$ is always a bounded and simply connected open set and $Q_c^n : \mathrm{Comp}(z, n, r) \to \mathbb{D}(Q_c^n(z), r)$ is always surjective and proper. When $c \in \mathbb{R}$ and $Q_c$ is considered as a map on the real line, let us denote the connected component of $Q_c^{-n}(\mathbb{I}(Q_c^n(x), r))$ containing $x$ by $\mathrm{comp}(x, n, r)$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$, $r > 0$. Contrary to the complex situation, in general $Q_c^n : \mathrm{comp}(x, n, r) \to \mathbb{I}(Q_c^n(x), r)$ fails to be surjective, when some critical value appears as extreme value in the range.

Given an open interval $I$, denote $\mathbb{C}_I := (\mathbb{C} \setminus \mathbb{R}) \cup I$. Given $x \in \mathbb{R}$ and $n \in \mathbb{N}$, if $Q_a$ is locally injective at $x$, i.e. $(Q_a^n)'(x) \neq 0$, let us denote its local inverse around $x$ by $\mathrm{Inv}(x, n)$. Moreover, for the maximal open interval $J$ containing $x$ on which $Q_a^n$ is injective, $\mathrm{Inv}(x, n)$ extends to a univalent function on $\mathbb{C}_I$, where $I := Q_c^n(J)$, and as a holomorphic function, we always consider the natural domain of $\mathrm{Inv}(x, n)$ as $\mathbb{C}_I$; as an interval map, we always consider the natural domain of $\mathrm{Inv}(x, n)$ as $I$.

Given a real number $x$, let $\lfloor x \rfloor$ be the maximal integer no larger than $x$ and let $\lceil x \rceil$ be the minimal integer no less than $x$.

Given real numbers $X_1, \cdots, X_n$, we denote their maximum and minimum by $X_1 \vee \cdots \vee X_n$ and $X_1 \wedge \cdots \wedge X_n$ respectively.

Give two variables $X$ and $Y$ of positive values, $X \asymp Y$ means that there exists a constant $C > 1$, at most depending only on $(a, b)$ and $\varphi$ appearing in (1.3), such that both $X \leq CY$ and $Y \leq CX$ hold simultaneously.

Given a real-valued or complex valued function $f$, let us denote its domain by $\mathrm{dom}(f)$, and given a subset $E$ of $\mathrm{dom}(f)$, denote the supremum norm of $f$ on $E$ by $\|f\|_E$.

Given a set $S$ in some topological space, denote its interior by $\mathrm{int}\, S$, its closure by $\mathrm{cl}\, S$

and its boundary by $\partial S$.

Given a set $S$ in some metric space, denote its diameter by $\mathrm{diam}(S)$.

The Lebesgue measure is denoted by Leb in general, and the Lebesgue measure on the real line is also denoted by $|\cdot|$ for short.

Let $\mathbb{N}$ denote the natural numbers starting from 1, i.e. $\mathbb{N} := \{1, 2, 3, \cdots\}$, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Chapter 2

# Preliminaries

## 2.1 Summary of useful facts about quadratic maps

Since both factors $Q_a$ and $Q_b$ in the skew-product construction of our dynamical system $F$ in (1.3) are quadratic maps, let us start with reviewing facts about quadratic maps that will be used in our argument. For background knowledge in real and complex one-dimensional dynamics that might be used, one may refer to, for example, [dMvS93] and [Mil06] respectively.

### 2.1.1 The quadratic family

For the quadratic family $\{Q_c : \mathbf{I}_{\beta_c} \circlearrowleft\}_{c\in(1,2]}$, let us introduce some conditions on the parameter $c$ as follows.

**Definition 2.1.1.** Given $c \in (1,2]$,

- $c$ is called **stochastic**, denoted by $c \in \mathscr{C}_{ST}$, if $Q_c$ admits an absolutely continuous invariant probability measure.

- $c$ is called **Collet-Eckmann**, denoted by $c \in \mathscr{C}_{CE}$, if

$$\liminf_{n\to\infty} \frac{1}{n} \log |(Q_c^n)'(c)| > 0. \tag{2.1}$$

- $c$ is called **Benedicks-Carleson**, denoted by $c \in \mathscr{C}_{BC}$, if $c \in \mathscr{C}_{CE}$ and

$$\liminf_{n\to\infty} \frac{\log |Q_c^n(0)|}{\sqrt{n}} > -\infty. \tag{2.2}$$

- $c$ is called **Misiurewicz-Thurston**, denoted by $c \in \mathscr{C}_{MT}$, if the critical orbit of $Q_c$ is pre-periodic but not periodic.

- $c$ is called **renormalizable**, if there exist $r \in (0, c)$ and $p \in \mathbb{N}$, such that

$$Q_c^p(\mathbf{I}_r) \subset \mathbf{I}_r\,,\ Q_c^p(\partial\mathbf{I}_r) \subset \partial\mathbf{I}_r \text{ and } \operatorname{int} Q_c^j(\mathbf{I}_r) \cap \operatorname{int} Q_c^k(\mathbf{I}_r) = \varnothing,\ 0 \le j < k < p. \tag{2.3}$$

  $\mathbf{I}_r$ satisfying (2.3) is called a **restrictive interval** of $Q_c$ and $p$ is called the **renormalization period** associated to $\mathbf{I}_r$.

- $c$ is called **infinitely renormalizable**, denoted by $c \in \mathscr{C}_{IR}$, if there are infinitely many renormalization periods of $Q_c$ . Otherwise, $c$ is called **at most finitely renormalizable**, denoted by $c \in \mathscr{C}_{FR}$.

For later usage, let us summarize some celebrated results together with some basic facts on the conditions above in the following theorem.

**Theorem 2.1.1.** *For the parameter sets introduced in Definition 2.1.1, we have:*

*(1)* $\mathscr{C}_{ST}$ *has positive measure and* $\mathscr{C}_{ST} \subset \mathscr{C}_{FR}$.

*(2)* $\mathscr{C}_{CE} \subset \mathscr{C}_{ST}$ *and* $\mathscr{C}_{CE}$ *has full measure in* $\mathscr{C}_{ST}$*; moreover, every Collet-Eckmann parameter in* $(1, 2)$ *is a Lebesgue density point of the following set:*

$$\left\{c \in \mathscr{C}_{CE} : \limsup_{n\to\infty} \frac{\log \frac{1}{|Q_c^n(0)|}}{\log n} = 1\right\}.$$

For $\mathscr{C}_{ST}$ of positive measure, see [Jak81] and [BC85]. For $\mathscr{C}_{ST} \subset \mathscr{C}_{FR}$, see, for example, [BL91]. For $\mathscr{C}_{CE} \subset \mathscr{C}_{ST}$, see [CE83] together with [Now85]. For the rest in assertion (2), see [AM05] and [GS13].

In view of that we will apply complex analysis argument to the base dynamics $Q_a$, it is convenient to consider the real quadratic family $\{Q_c : c \in (1, 2]\}$ as a subfamily of the complex quadratic family $\{Q_c : c \in \mathbf{D}_2\}$. For the complex family, we have the following elementary fact that will be used in § 3.1.

**Fact 2.1.2.** *Given $r > 2$, for every $c \in \mathbf{D}_2$, $\mathbb{C} \setminus \mathbf{D}_r$ is forward $Q_c$-invariant and is contained in the attracting basin of $\infty$. More straightforwardly,*

$$Q_c(\mathbb{C} \setminus \mathbf{D}_r) \subset \mathbb{C} \setminus \mathbf{D}_{r^2-|c|} \subset \mathbb{C} \setminus \mathbf{D}_{r(r-1)}.$$

*Moreover, there exists a constant $C_r > 1$, such that for every $c \in \mathbf{D}_2$, every $n \in \mathbb{N}$ and every $z = Q_c^n(w)$ with $w \in \mathbb{C} \setminus \mathbb{D}_r$, we have:*

$$C_r^{-1} \cdot |z|^{2^{-n}} \le |w| \le C_r \cdot |z|^{2^{-n}} \quad \textit{and} \quad C_r^{-1} \cdot 2^n \, |z|^{1-2^{-n}} \le |(Q_c^n)'(w)| \le C_r \cdot 2^n \, |z|^{1-2^{-n}}. \quad (2.4)$$

*Proof.* Only (2.4) needs verification. Consider the Böttcher map $\psi_c$ of $Q_c$ around infinity, which is well defined and univalent on $\mathbb{C} \setminus \mathbf{D}_2$ when $c \in \mathbf{D}_2$, as follows:

$$\lim_{z\to\infty} \frac{\psi_c(z)}{z} = 1 \qquad \text{and} \qquad \psi_c \circ Q_c = Q_0 \circ \psi_c \quad \text{on} \quad \mathbb{C} \setminus \mathbf{D}_2. \qquad (2.5)$$

Noting that both of the families $\{\frac{\psi_c(z)}{z}\}_{c\in\mathbf{D}_2}$ and $\{\psi_c'(z)\}_{c\in\mathbf{D}_2}$ are uniformly bounded on $\mathbb{C} \setminus \mathbf{D}_r$, the conclusion in (2.4) follows from conjugate relation between $Q_c$ and $Q_0$ in (2.5). □

### 2.1.2 About renormalization

Let us list some well known facts about the attractor of a stochastic map in Lemma 2.1.3 below for further usage. See, for example, [BL91] for reference. Recall that $\mathscr{C}_{ST} \subset \mathscr{C}_{FR}$. Ahead of the statement of Lemma 2.1.3, given $c \in \mathscr{C}_{FR}$, we introduce some notations for $Q_c$ . Let $\hat{I}_c$ denote the minimal restrictive interval of $Q_c$ and let $\hat{p}_c$ denote the associated maximal renormalization period(when $Q_c$ is non-renormalizable, $\hat{I}_c = \mathbf{I}_{\beta_c}$ and $\hat{p}_c = 1$ by definition). Moreover, denote

$$\check{I}_c := [Q_c^{2\hat{p}_c}(0), Q_c^{\hat{p}_c}(0)] \subset \hat{I}_c \qquad \text{and} \qquad \check{I}_c^k := Q_c^k(\check{I}_c)\,,\ k \in \mathbb{N}_0. \tag{2.6}$$

Note that $Q_c^{\hat{p}_c} : \hat{I}_c \circlearrowleft$ is always an S-unimodal map, and $\check{I}_c \subset \operatorname{int} \hat{I}_c$ if and only if $c \notin \mathscr{C}_{MT}$.

Following the notations above, we have:

**Lemma 2.1.3.** *Given $c \in \mathscr{C}_{ST}$, the following statements hold.*

*(1) $Q_c$ admits a unique ergodic a.c.i.p. $\mu_c$ with* $\operatorname{supp}(\mu_c) = \bigcup_{k=0}^{\hat{p}_c - 1} \check{I}_c^k$*, and $Q_c$ is topologically transitive on* $\operatorname{supp}(\mu_c)$.

*(2) $Q_c^{\hat{p}_c}(\check{I}_c) = \check{I}_c$ and $Q_c^{\hat{p}_c} : \check{I}_c \circlearrowleft$ is topologically exact.*

To prove the Main Theorem, the parameter $a$ in $F_{a,b,\alpha}$ will be chosen from $\mathscr{C}_{BC}$, which is of positive measure according to Theorem 2.1.1. Once $a$ is chosen, to obtain the corresponding range of the parameter $b$, let us introduce the following notions.

**Definition 2.1.2.** Given $R > 1$, we say that $c \in (1, 2)$ is ***R*-dominated**, denoted by $c \in \mathscr{C}_{DM}(R)$, if it satisfies the following assumptions.

- There exists a compact set $\Lambda_c \subset \mathbf{I}_{\beta_c}$, such that $0 \in \operatorname{int} \Lambda_c$ and $Q_c(\Lambda_c) \subset \operatorname{int} \Lambda_c$.
- There exist $p_c \in \mathbb{N}$ and $R_c \in (1, R)$, such that $\|(Q_c^p)'\|_{\Lambda_c} \leq R_c^{p_c}$ for $1 \leq p \leq p_c$.

Moreover, define

$$\mathscr{B}(R) := \left\{ c \in \mathscr{C}_{DM}(R) \cap \mathscr{C}_{CE} : \lim_{n\to\infty} \frac{\log |Q_c^n(0)|}{n} = 0 \right\}.$$

*Remark.* According to [BST03, Lemma 3.1], $\mathscr{C}_{DM}(2) = (1, 2)$.

We have to show that, as the range of parameter $b$ for given $a$, $\mathscr{B}(R)$ is of positive measure for every $R > 1$. Thanks to Sullivan's theory on renormalization of quadratic-like maps in [Sul92] and Theorem 2.1.1, we have:

**Lemma 2.1.4.** *For every $R > 1$, $\mathscr{C}_{DM}(R)$ is a nonempty open subset of $(1, 2)$ and $\mathscr{B}(R)$ has positive measure.*

*Proof.* $\mathscr{C}_{DM}(R)$ is open by definition and simple continuity argument. Let us show that it is nonempty. According to [Sul92, Theorem 1], the renormalization in the quadratic family has the so called "beau" property, which in particular implies that there exist $n_0 \in \mathbb{N}$ and $C > 1$, such that if $Q_c$ is $n$-times renormalizable for some $n \geq n_0$, then for the restrictive interval $\mathbf{I}_{r_n}$ of the $n$-th renormalization of $Q_c$ and the associated renormalization period $p_n \in \mathbb{N}$, we have the following distortion control:

$$\sup_{x_1,x_2\in Q_c^i(\mathbf{I}_{r_n})} \frac{|(Q_c^j)'(x_1)|}{|(Q_c^j)'(x_2)|} \leq C, \quad 1 \leq i < i + j \leq p_n. \tag{2.7}$$

To proceed, denote

$$I_k := Q_c^k(\mathbf{I}_{r_n}),\ 0 \leq k \leq p_n \quad \text{and} \quad \widehat{\Lambda}_c := \bigcup_{k=0}^{p_n-1} Q_c^k(I_k),$$

and let us use (2.7) to estimate $\|(Q_c^{p_n})'\|_{\widehat{\Lambda}_c}$ first. By definition, $|I_1| = r_n^2$, $|I_{p_n}| \leq |I_0| = 2r_n$ and $Q_c^{p_n-1}$ is injective on $I_1$. Then due to (2.7),

$$C^{-1}\|(Q_c^{p_n-1})'\|_{I_1} \cdot |I_1| \leq |I_{p_n}| \Longrightarrow \|(Q_c^{p_n-1})'\|_{I_1} \cdot r_n \leq 2C.$$

Since $Q_c$ acts cyclically on $\{I_k\}_{k=0}^{p_n-1}$, by chain rule, (2.7) and the inequality above,

$$\|(Q_c^{p_n})'\|_{I_k} \leq C \cdot \|(Q_c^{p_n-1})'\|_{I_1} \cdot \|Q_c'\|_{I_0} \leq 4C^2, \quad 0 \leq k < p_n.$$

It follows that

$$\|(Q_c^{p_n})'\|_{\widehat{\Lambda}_c} \leq 4C^2 \Longrightarrow \|(Q_c^{kp_n+l})'\|_{\widehat{\Lambda}_c} < (4C^2)^k \cdot 4^l, \quad \forall\, k \in \mathbb{N}_0, 0 \leq l < p_n.$$

As a result, given $R > 1$, by assuming that $n$ is so large that $R^{p_n} > 4C^2$, for an arbitrarily chosen $R_c \in ((4C^2)^{1/p_n}, R)$, there exists $k \in \mathbb{N}$, such that for $p_c = kp_n$,

$$\|(Q_c^p)'\|_{\widehat{\Lambda}_c} \leq \frac{1}{2} R_c^{p_c}, \quad 1 \leq p \leq p_c. \tag{2.8}$$

Fixing such a pair of $(R_c\,, p_c)$, we still need to show the existence of $\Lambda_c$. To satisfy the fist assumption in Definition 2.1.2, we may further assume that $Q_c^{p_n}(0) \in \operatorname{int} I_0$, which can be realized at least when we require that $c$ is $(n+1)$-times renormalizable. Then since either $r_n$ or $-r_n$ is a repelling hyperbolic fixed point of $Q_c^{p_n}$, when $\tilde{r}_n > 0$ is slightly smaller than $r_n$, $Q_c^{p_n}(\mathbf{I}_{\tilde{r}_n}) \subset \mathbb{I}_{\tilde{r}_n}$. Therefore we can find a sequence of closed intervals $J_0, J_1, \cdots, J_{p_n}$, such that $J_0 = J_{p_n} = \mathbf{I}_{\tilde{r}_n}$ and $Q_c(J_p) \subset \operatorname{int} J_{p+1}$ for $0 \leq p < p_n$. Then for $\Lambda_c := \cup_{p=0}^{p_n-1} J_p$, the first assumption in Definition 2.1.2 holds. Since the difference between $J_p$ and $I_p$ can be chosen arbitrarily small for $0 \leq p < p_n$, the second assumption in Definition 2.1.2 can be verified by (2.8) and continuity. That is to say, $\mathscr{C}_{DM}(R)$ is nonempty.

It remains to prove that $\mathscr{B}(R)$ has positive measure. The discussion in the last paragraph shows that, in particular, an infinitely renormalizable parameter $c$ is always in $\mathscr{C}_{DM}(R)$ with appropriate choice of $\Lambda_c$, $p_c$ and $R_c$. Since $\mathscr{C}_{DM}(R)$ is open and since infinitely renormalizable parameters are in the closure of Misiurewicz-Thurston parameters, there exist a Misiurewicz-Thurston $Q_c$ and $r > 0$, such that $\mathbb{I}(c, r) \subset \mathscr{C}_{DM}(R)$. Since $c \in \mathscr{C}_{MT} \subset \mathscr{C}_{CE}$, the conclusion that $\mathscr{B}(R)$ has positive measure follows from the second assertion in Theorem 2.1.1. □

### 2.1.3 Collet-Eckmann condition

When the Collet-Eckmann condition is imposed on the real quadratic map $Q_c$ for some $c \in (1,2)$, there are celebrated theorems on equivalent conditions, either when $Q_c$ is treated as an S-unimodal map on $\mathbf{I}_{\beta_c}$, see [NS98] for example; or when it is treated as a holomorphic map on the Riemann sphere with a unique critical point in the Julia set, see [PRLS03] for example. We will not make full use of these equivalent relations, but list some of them in the following theorem for later usage.

**Theorem 2.1.5.** *For every $c \in \mathscr{C}_{CE}$, there exist $K_c > 0$, $\lambda_c > 1$ and $\hat{r}_c \in (0,1)$, such that the following statements hold.*

*(1) When $Q_c$ is considered as a real map on $\mathbb{R}$,*

$$|(Q_c^n)'(x)| \geq K_c \lambda_c^n \cdot \left(|x| \wedge |Q_c(x)| \wedge \cdots \wedge |Q_c^{n-1}(x)|\right), \quad \forall\, x \in \mathbb{R},\ n \in \mathbb{N}. \tag{2.9}$$

*(2) When $Q_c$ is considered as a complex map on $\mathbb{C}$,*

$$\operatorname{diam}\left(\operatorname{Comp}(z, n, \hat{r}_c)\right) \leq \lambda_c^{-n}, \quad \forall\, z \in \mathbb{C},\ n \in \mathbb{N}. \tag{2.10}$$

*Remark.* (2.9) is known as **semi-uniformly expanding** property, see [NS98]. (2.10) is known as **exponential shrinking** property, see [PRLS03]. Note that although in [PRLS03], (2.10) is stated for $z$ in the Julia set, we can get rid of this restriction here, thanks to the fact that the Fatou set of $Q_c$ coincides with the basin of attraction of infinity. In our argument, (2.9) will be applied to $Q_b$ in § 2.3 and (2.10) will mainly be applied to $Q_a$ for $z \in \mathbf{I}_{\beta_a}$.

Let us give some useful properties as direct corollary of (2.9) or (2.10) below, which themselves are also well known. We do not bother to give the most precise form of their statements for simplicity.

**Corollary 2.1.6.** *For every $c \in \mathscr{C}_{CE}$, the following statements hold.*

*(1) Given $\delta > 0$ and $\tau \geq 1$, for every $x \in \mathbb{R}$,*

$$x, Q_c(x), \cdots, Q_c^{n-1}(x) \notin \mathbb{I}_\delta \ \textit{and}\ Q_c^n(x) \in \mathbb{I}_{\tau\delta} \Longrightarrow |(Q_c^n)'(x)| \geq \frac{K_c}{2\tau}\lambda_c^n. \tag{2.11}$$

*(2) Given $z \in \mathbb{C}$, $n \in \mathbb{N}$ and $r \in (0, \hat{r}_c]$,*

$$Q_c^n \ \textit{is injective on}\ \mathrm{Comp}(z, n, r) \Longrightarrow |(Q_c^n)'(z)| \geq \lambda_c^n \cdot r. \tag{2.12}$$

*(3) Given $z \in \mathbb{C} \setminus \mathbb{D}_3$, $n \in \mathbb{N}$ and $w \in Q_c^{-n}(z) \cap \mathbb{D}_3$, there exists a unique $1 \leq k \leq n$, such that $|Q_c^i(w)| < 3$ when $0 \leq i < k$ and $|Q_c^i(w)| \geq 3$ when $k \leq i \leq n$, and*

$$|(Q_c^n)'(w)| \geq C\,\hat{r}_c \cdot 2^{n-k}\lambda_c^k\,|z|^{1-2^{k-n}}, \tag{2.13}$$

*where $C = C_3^{-1}$ for $C_3$ appearing in (2.4) with $r = 3$.*

*(4) Let $E$ be a connected subset of $\mathbb{C}$ with $E \cap \mathbf{I}_{\beta_c} \neq \emptyset$. Then for every $n \in \mathbb{N}$, we have:*

$$\mathrm{diam}(Q_c^n(E)) < \hat{r}_c \Longrightarrow \lambda_c^{n-1} \cdot \mathrm{diam}(E) \leq \left(\frac{\mathrm{diam}(Q_c^n(E))}{\hat{r}_c}\right)^{\frac{\log \lambda_c}{\log 6}}. \tag{2.14}$$

*Proof.*

(1) By continuity, we may assume that $Q_c^n(x) \neq 0$. Then due to (2.9),

$$2|Q_c^n(x)| \cdot |(Q_c^n)'(x)| = |(Q_c^{n+1})'(x)| \geq K_c\lambda_c^{n+1} \cdot (\delta \wedge |Q_c^n(x)|).$$

Dividing both sides of the inequality above by $2|Q_c^n(x)|$, (2.11) follows.

(2) By the assumption in (2.12), $\tau := \mathrm{Inv}(z, n)$ is well defined on $\mathbb{D}(w, r)$ for $w := Q_c^n(z)$, and according to (2.10), $\tau(\mathbb{D}(w, r)) \subset \mathbb{D}(z, \lambda_c^{-n})$. Then by Schwarz lemma, $|\tau'(w)| \leq (\lambda_c^n r)^{-1}$, which proves (2.12).

(3) Both the existence and uniqueness of $k$ follow from the forward $Q_c$-invariance of $\mathbb{C} \setminus \mathbb{D}_3$. Since $|Q_c^k(w)| \geq 3$, on the one hand, $Q_c^k$ is injective on $\mathrm{Comp}(w, k, \hat{r}_c)$, so by (2.12), $|(Q_c^k)'(w)| \geq \lambda_c^k \hat{r}_c$; on the other hand, by (2.4) for $r = 3$, $|(Q_c^{n-k})'(Q_c^k(w))| \geq C_3^{-1} 2^{n-k} |z|^{1-2^{k-n}}$. Combining these two inequalities yields (2.13).

(4) Fix $x_0 \in E \cap \mathbf{I}_{\beta_c}$ and let $k \in \mathbb{N}_0$ be the maximal number such that $\mathrm{diam}(Q_c^{n+k}(E)) < \hat{r}_c$. On the one hand, from (2.10) and the connectedness of $E$ we know that

$$E \subset \mathrm{Comp}(x_0, n + k, \hat{r}_c) \Longrightarrow \mathrm{diam}(E) \leq \lambda_c^{-n-k}.$$

On the other hand, from $Q_c^{n+i}(E) \subset \mathbb{D}(Q_c^{n+i}(x_0), \hat{r}_c) \subset \mathbb{D}_3$ for $0 \leq i \leq k$ we know that

$$\hat{r}_c \leq \mathrm{diam}(Q_c^{n+k+1}(E)) \leq 6^{k+1} \cdot \mathrm{diam}(Q_c^n(E)).$$

Combining the two displayed inequalities above, (2.14) follows.

□

## 2.2 Elementary properties about iteration of $F$

### 2.2.1 Specification of parameter choices

Now we are ready to specify the parameter set $\mathscr{P}$ of $(a, b)$ in the Main Theorem. Recall the constant $\lambda_c > 1$ introduced in Theorem 2.1.5 for $c \in \mathscr{C}_{CE}$ and the Borel subset $\mathscr{C}_{BC} \subset$

$\mathscr{C}_{CE}$ defined in Definition 2.1.1. For clarity, we will fix the choice of $\lambda_a$ for each $a \in \mathscr{C}_{BC}$ in such a way that $a \mapsto \lambda_a$ defines a Borel measurable function on $\mathscr{C}_{BC}$. Also recall the parameter set $\mathscr{B}(R)$ introduced in Definition 2.1.2 for every $R > 1$. Then we define

$$\mathscr{P} := \{(a, b) \in (1, 2] \times (1, 2) : a \in \mathscr{C}_{BC}, b \in \mathscr{B}(\lambda_a)\}. \tag{2.15}$$

*Remark.* Note that $\mathscr{P}$ can be written as

$$\bigcup_{R \in \mathbb{Q} \cap (1,2)} \{a \in \mathscr{C}_{BC} : \lambda_a \geq R\} \times \mathscr{B}(R),$$

where each component in the countable union above is a Borel subset of $\mathbb{R}^2$ with positive two dimensional Lebesgue measure. Therefore, $\mathscr{P}$ is also a Borel subset of $\mathbb{R}^2$ with positive two dimensional Lebesgue measure.

From now on let us fix a pair of parameters $(a, b) \in \mathscr{P}$ and drop all the subscripts of $F_{a,b,\alpha}$ defined in (1.3) for convenience. Moreover, for $F^n$, the $n$-times iteration of $F$, let us denote its second coordinate by $f_n$, i.e.

$$F^n(x, y) = (Q_a^n(x), f_n(x, y)) , \quad f_n(x, y) = Q_b(f_{n-1}(x, y)) + \alpha \cdot \varphi(Q_a^{n-1}(x)), \quad \forall\, n \in \mathbb{N}.$$

Without loss of generality, the coupling function $\varphi$ can be normalized as below for convenience:

$$\|\varphi\|_{\mathbb{D}_3} \leq 1 \qquad \text{and} \qquad |\varphi(z)| \leq |z|^{d_\varphi} \text{ on } \mathbb{C} \setminus \mathbb{D}_1 \quad \text{for} \quad d_\varphi := \deg \varphi. \tag{2.16}$$

Recall the $Q_b$-invariant set $\Lambda_b \subset \mathbf{I}_{\beta_b}$ introduced in Definition 2.1.2 for $c = b$. Since $Q_b(\Lambda_b) \subset \operatorname{int} \Lambda_b$, by supposing that $\alpha > 0$ is small enough, namely $\alpha$ does not exceed the distance between $Q_b(\Lambda_b)$ and $\partial\Lambda_b$, we can guarantee that $\mathbf{I}_{\beta_a} \times \Lambda_b$ is forward $F$-invariant. Also note that since $Q_b$ is Collet-Eckmann, according to (2.9), there exists $q_b \in \mathbb{N}$, such that

$$y \in \mathbb{R},\ Q_b^k(y) \notin Q_b(\Lambda_b),\ k = 0, \cdots, q_b - 1 \Longrightarrow |(Q_b^{q_b})'(y)| \geq 3.$$

Then by continuity, when $\alpha > 0$ is small,

$$(x, y) \in \mathbf{I}_{\beta_a} \times \mathbb{R},\ f_k(x, y) \notin \Lambda_b,\ k = 0, \cdots, q_b - 1 \Longrightarrow |\partial_y f_{q_b}(x, y)| \geq 2.$$

That is to say, the iteration of $F$ outside $\mathbf{I}_{\beta_a} \times \Lambda_b$ is uniform expanding in the vertical direction, and hence less interesting in our discussion. As a result, we will only focus on the dynamical system

$$F : \mathbf{I}_{\beta_a} \times \Lambda_b \circlearrowleft .$$

For preparation of the proof of the main theorem, in the rest of this section we give a priori estimates on iteration of $F$ under both real and complex settings.

### 2.2.2 Iteration of $F$ in the real domain

To view the vertical direction of $F$ as a random perturbation of $Q_b$, using elementary calculation we can obtain some rough estimates about the iteration of $F$, which will be used in several occasions in our argument.

Let us start with some notations. Given $\delta > 0$, $x \in \mathbb{R}$ and $r > 0$, denote

$$B_\delta^+(\mathbf{I}(x, r)) := \mathbf{I}(x, r + \delta) \quad \text{and} \quad B_\delta^-(\mathbf{I}(x, r)) := \mathbf{I}(x, (r - \delta) \vee 0).$$

By definition, for any closed subinterval $J$ of $\Lambda_b$,

$$B^-_\alpha(Q_b(J)) \subset f_1(\{x\} \times J) \subset f_1(\mathbf{I}_{\beta_a} \times J) \subset B^+_\alpha(Q_b(J)) \subset \Lambda_b, \quad \forall\, x \in \mathbf{I}_{\beta_a}.$$

Now given a closed subinterval $J$ of $\Lambda_b$, define $J^+_0 = J^-_0 = J$ and for $n \in \mathbb{N}_0$, define $J^+_{n+1} = B^+_\alpha(Q_b(J^+_n))$ and $J^-_{n+1} = B^-_\alpha(Q_b(J^-_n))$ inductively. By definition and induction,

$$J^-_n \subset f_n(\{x\} \times J) \subset f_n(\mathbf{I}_{\beta_a} \times J) \subset J^+_n \quad \text{and} \quad J^-_n \subset Q^n_b(J) \subset J^+_n, \quad \forall\, n \in \mathbb{N}_0,\ x \in \mathbf{I}_{\beta_a}.$$

By definition and intermediate value theorem,

$$|J^+_{n+1}| = D^+_n \cdot |J^+_n| + 2\alpha, \text{ where } D^+_n = |Q'_b(y_n)| \text{ for some } y_n \in J^+_n, \quad \forall\, n \in \mathbb{N}_0,$$

and then by induction,

$$|J^+_{n+1}| = |J| \cdot \prod_{k=0}^{n} D^+_k + 2\alpha \cdot (1 + \sum_{k=1}^{n} \prod_{j=k}^{n} D^+_j), \quad \forall\, n \in \mathbb{N}_0. \tag{2.17}$$

Similarly,

$$|J^-_{n+1}| \geq D^-_n \cdot |J^-_n| - 2\alpha, \text{ where } D^-_n = \min_{y \in J^-_n} |Q'_b(y)|,$$

and therefore

$$|J^-_{n+1}| \geq |J| \cdot \prod_{k=0}^{n} D^-_k - 2\alpha \cdot (1 + \sum_{k=1}^{n} \prod_{j=k}^{n} D^-_j), \quad \forall\, n \in \mathbb{N}_0. \tag{2.18}$$

Since $|Q'_b| \leq 4$ on $\Lambda_b$, from (2.17) it follows that for every subinterval $J$ of $\Lambda_b$,

$$|f_n(\mathbf{I}_{\beta_a} \times J)| \leq |J^+_n| < 4^n(|J| + \frac{2}{3}\alpha), \quad \forall\, n \in \mathbb{N}. \tag{2.19}$$

In particular, taking $J = \{y\}$, the inequality above implies that

$$|f_n(x,y) - Q_b^n(y)| < \frac{2}{3} \cdot 4^n \alpha, \quad \forall\, n \in \mathbb{N},\ x \in \mathbf{I}_{\beta_a},\ y \in \Lambda_b.$$

As a result, given $x \in \mathbf{I}_{\beta_a}$, $y_1, y_2 \in \Lambda_b$ and $n \in \mathbb{N}$,

$$|f_n(\{x\} \times [y_1, y_2])| - |Q_b^n(y_1) - Q_b^n(y_2)| \geq -\sum_{i=1}^{2} |f_n(x, y_i) - Q_b^n(y_i)| > -\frac{4^{n+1}}{3} \cdot \alpha.$$

Note that given an arbitrary closed interval $J$, for every $n \in \mathbb{N}$, there exist $y_1, y_2 \in J$, such that $Q_b^n(J) = Q_b^n([y_1, y_2])$. It follows that when $J \subset \Lambda_b$,

$$|f_n(\{x\} \times J)| > |Q_b^n(J)| - \frac{4^{n+1}}{3} \cdot \alpha, \qquad \forall\, n \in \mathbb{N},\ x \in \mathbf{I}_{\beta_a}. \tag{2.20}$$

Besides the estimates above, since the derivative of a quadratic map is linear, we will frequently use the following basic fact to control the distortion for pseudo orbit of $Q_b$.

**Fact 2.2.1.** *Let $\{x_i\}_{i=1}^n$ and $\{\tilde{x}_i\}_{i=1}^n$ be two sequences of nonzero real numbers. Then*

$$\sum_{i=1}^{n} \frac{|x_i - \tilde{x}_i|}{|x_i|} \leq \theta < 1 \Longrightarrow e^{-\frac{\theta}{1-\theta}} \leq \frac{|\prod_{i=1}^n \tilde{x}_i|}{|\prod_{i=1}^n x_i|} \leq e^{\frac{\theta}{1-\theta}}. \tag{2.21}$$

*Proof.* Note that when $|t| \leq \theta < 1$, $|\log(1+t)| \leq \log \frac{1}{1-|t|} = \int_0^{|t|} \frac{ds}{1-s} \leq \frac{|t|}{1-\theta}$. As a result,

$$\left|\log \frac{|\prod_{i=1}^n \tilde{x}_i|}{|\prod_{i=1}^n x_i|}\right| = \left|\sum_{i=1}^{n} \log\left|1 + \frac{\tilde{x}_i - x_i}{x_i}\right|\right| \leq (1-\theta)^{-1} \cdot \sum_{i=1}^{n} \frac{|x_i - \tilde{x}_i|}{|x_i|} \leq \frac{\theta}{1-\theta}.$$

The conclusion follows. □

### 2.2.3 Iteration of $F$ in the complex domain

To provide a little more precise estimates about iteration of $F$, we have to take the partial hyperbolicity of $F$ into consideration. Restricted to $\mathbf{I}_{\beta_a} \times \Lambda_b$, by considering $F$ as a perturbation of $Q_a \times Q_b$, the partial hyperbolicity of $F$ is automatically inherited from the domination relation between $Q_a$ and $Q_b$, namely $b \in \mathscr{B}(\lambda_a)$, provided that $\alpha > 0$ is small. This fact is explicitly indicated in Lemma 2.2.2, where we view $F$ as a holomorphic map on $\mathbb{C} \times \mathbb{C}$ for the sake of using complex analysis arguments in Chapter 3.

Recall that $R_b < \lambda_a$ by our assumption and let us fix a constant $\hat{R}_b \in (R_b, \lambda_a)$, and to be definite, let

$$\hat{R}_b := \sqrt{\lambda_a R_b} \in (R_b, \lambda_a).$$

Then by continuity, there exist $A_0 \geq 1$ and $\delta_b > 0$, such that when $\alpha > 0$ is small enough, we have:

$$|\partial_x f_n| \leq A_0 \alpha \qquad \text{and} \qquad |\partial_y f_n| \leq \hat{R}_b^{p_b} \quad \text{on} \quad \mathbb{D}_3 \times \mathbb{D}(\Lambda_b, 2\delta_b), \quad 1 \leq n \leq p_b. \tag{2.22}$$

Here $\mathbb{D}(\Lambda_b, 2\delta_b) := \cup_{x \in \Lambda_b} \mathbb{D}(x, 2\delta_b)$. A further continuity argument implies that:

**Lemma 2.2.2.** *The following statement holds when $\alpha > 0$ is small. Given $x_0 \in \mathbf{I}_{\beta_a}$, $y_0 \in \Lambda_b$ and $n \in \mathbb{N}$, denote $(x_k, y_k) = F^k(x_0, y_0)$ for $0 \leq k \leq n$. Then for*

$$\hat{A}_0 := \frac{A_0}{\lambda_a^{p_b} - \hat{R}_b^{p_b}}, \quad U := \mathrm{Comp}(x_0, n, \hat{r}_a) \quad \textit{and} \quad V_r := U \times \mathbb{D}(y_0, r), \;\; 0 < r \leq \hat{R}_b^{-n-p_b} \delta_b\,,$$

*we have*

$$f_k(V_r) \subset \mathbb{D}\left(y_k, \hat{A}_0 \lambda_a^{k-n} \alpha + \hat{R}_b^{k+p_b} r\right) \subset \mathbb{D}(y_k, 2\delta_b), \quad 0 \leq k \leq n, \tag{2.23}$$

*and consequently*

$$\|\partial_y f_j\|_{F^k(V_r)} \leq \hat{R}_b^{j+p_b}, \quad 1 \leq j+k \leq n. \tag{2.24}$$

*Proof.* Denote $\Omega := \mathbb{D}_3 \times \mathbb{D}(\Lambda_b, 2\delta_b)$. Once (2.23) has been proved, then in (2.24), $F^k(V_r) \subset \Omega$. Therefore, according to the second inequality in (2.22), when $j \leq (n-k) \wedge p_b$, (2.24) is trivial; when $p_b \leq j \leq n-k$,

$$\|\partial_y f_j\|_{F^k(V_r)} \leq \hat{R}_b^{p_b} \cdot \|\partial_y f_{j-p_b}\|_{F^{k+p_b}(V_r)} \quad \text{and} \quad F^{k+p_b}(V_r) \subset \Omega,$$

so (2.24) follows from a simple induction argument.

To prove (2.23), first note that since $\mathbb{D}(x, \hat{r}_a) \subset \mathbb{D}_3$ and since $Q_a^{-1}(\mathbb{D}_3) \subset \mathbb{D}_3$, $Q_a^k(U) \subset \mathbb{D}_3$ for $0 \leq k \leq n$. As a result, when $k \leq p_b \wedge n$, (2.23) follows from (2.22), so we may assume that $n > p_b$.

Given $(z_0, w_0) \in V_r$, denote $(z_k, w_k) = F^k(z_0, w_0)$ and $\Delta_k = |y_k - w_k|$ for $0 \leq k \leq n$. Note that by (2.10) and definition, $z_k \in Q_a^k(U) \subset \mathbb{D}(x_k, \lambda_a^{k-n})$. If $\Delta_k \leq \delta_b$ has been proved for some $k \leq n - p_b$, then from (2.22) we know that

$$\Delta_{k+p_b} \leq \|\partial_x f_{p_b}\|_\Omega \cdot |x_k - z_k| + \|\partial_y f_{p_b}\|_\Omega \cdot \Delta_k \leq A_0 \alpha \cdot |x_k - z_k| + \hat{R}_b^{p_b} \Delta_k \leq A_0 \lambda_a^{k-n} \alpha + \hat{R}_b^{p_b} \Delta_k.$$

Now we can complete the proof of (2.23) by applying the inequality above inductively. Given $l \geq 1$ and $0 \leq k < p_b$ with $k' := lp_b + k \leq n$, once $\Delta_j \leq \delta_b$ has been proved for $j \leq k' - p_b$, then we have

$$\Delta_{k'} \leq A_0 \lambda_a^{k'-p_b-n} \alpha + \hat{R}_b^{p_b} \Delta_{k'-p_b} \leq \cdots \leq A_0 \lambda_a^{k'-p_b-n} \alpha \cdot \sum_{j=0}^{l-1} \left( \hat{R}_b \lambda_a^{-1} \right)^{jp_b} + \hat{R}_b^{lp_b} \Delta_k.$$

By the choice of $\hat{A}_0$ and the assumption on $r$, provided that $\hat{A}_0\alpha < \delta_b$, it follows that

$$\Delta_{k'} \leq \hat{A}_0 \lambda_a^{k'-n}\alpha + \hat{R}_b^{k'+p_b} r < 2\delta_b,$$

which completes the induction.

□

As a supplement of the lemma above, when the orbit $\{F^n(x,y)\}_{n\geq 1}$ is unbounded, we need the following control of the growth of $f_n$.

**Fact 2.2.3.** *When $\alpha \leq 1$, the following statement holds. Given $x_0, y_0 \in \mathbb{D}_3$, denote $(x_k, y_k) = F^k(x_0, y_0)$ for every $k \in \mathbb{N}$. Then we have:*

$$|x_k| < 2^{2^{k+1}} - 1 \quad \textit{and} \quad |y_k| < 2^{d_\varphi \cdot 2^{k+1}} - 1.$$

*Moreover, if $\mathbb{D}(y_0, r) \subset \mathbb{D}_3$ for some $r \geq \alpha$, then for every $(x, y) \in \mathbb{D}_3 \times \mathbb{D}(y_0, r)$,*

$$|f_k(x,y) - y_k| < 2^{d_\varphi \cdot 2^{k+2}} \cdot r.$$

*Proof.* All the estimates follow from induction on $k$ as below. It is easy to check that all the inequalities hold for $k = 0$. Then by induction,

$$|x_{k+1}| \leq |x_k|^2 + 2 < (2^{2^{k+1}} - 1)^2 + 2 \leq 2^{2^{k+2}} - 5,$$

and using the normalization condition imposed on $\varphi$ in (2.16),

$$|y_{k+1}| \leq |y_k|^2 + 2 + \alpha \cdot \max\{1, |x_k|^d\} \leq (2^{d_\varphi \cdot 2^{k+1}} - 1)^2 + 2 + (2^{2^{k+1}} - 1)^{d_\varphi} \leq 2^{d_\varphi \cdot 2^{k+2}} - 2.$$

For the second assertion in the lemma, note that since $x, y \in \mathbb{D}_3$, $|f_k(x,y)| \leq 2^{d_\varphi \cdot 2^{k+1}} - 1$.

Then by induction,

$$\begin{aligned}|f_{k+1}(x,y)-y_{k+1}| &\leq |f_k(x,y)+y_k|\cdot|f_k(x,y)-y_k|+\alpha\cdot|\varphi(Q_a^k(x))-\varphi(x_k)|\\ &< 2\cdot(2^{d_\varphi\cdot 2^{k+1}}-1)\cdot|f_k(x,y)-y_k|+2\alpha\cdot(2^{2^{k+1}}-1)^{d_\varphi}\\ &< 2^{d_\varphi\cdot 2^{k+3}}\cdot r.\end{aligned}$$

□

## 2.3 Building vertical expansion

In this section, we will use the assumption $b \in \mathscr{C}_{CE}$ and the sub-exponential recurrence assumption $\lim\limits_{n\to\infty}\frac{\log|Q_b^n(0)|}{n} = 0$ to establish vertical expansion for iteration of $F$ when the orbit keeps away from $\mathbb{I}_{\sqrt{\alpha}}$ in the vertical direction. All the results are summarized in Lemma 2.3.1, which are actually statements about random perturbation of $Q_b$. This kind of results are well known. For example, for the Misiurewicz-Thurston case of $Q_b$, see [Via97, § 2.2]; for similar setting on $Q_b$ to ours, see [BV96, § 3]. Our results are essentially contained in their arguments; however, since there is some difference between details in the statements, let us provide a self-contained proof here.

**Lemma 2.3.1.** *There exist constants $C_b > 0$ and $\sigma_b \in (1, \lambda_b)$, such that for any $\eta \in (0, \frac{1}{2})$, the following statements hold when $\alpha > 0$ is small. Given $(x_0, y_0) \in \mathbf{I}_{\beta_a} \times \Lambda_b$, denote $z_k := (x_k, y_k) := F^k(x_0, y_0)$ for $k \in \mathbb{N}_0$. Then we have:*

*(1) If $|y_0| < 2\sqrt{\alpha}$, then $|y_k| > \alpha^\eta$, $k = 1, \cdots, N_\alpha - 1$ and $|\partial_y f_{N_\alpha}(z_0)| \geq |y_0|\alpha^{-1+\eta}$. Here $N_\alpha \in \mathbb{N}$ is independent of $(x_0, y_0)$ and it satisfies $\sigma_b^{N_\alpha} \leq \alpha^{-1} \leq 4^{N_\alpha}$.*

*(2) Given $n \in \mathbb{N}$, if $|y_k| \geq \sqrt{\alpha}$ for $k = 0, 1 \cdots, n-1$, then $|\partial_y f_n(z_0)| \geq C_b\sqrt{\alpha}\sigma_b^n$. If, in addition, $|y_n| \leq \alpha^\eta$, then $|\partial_y f_n(z_0)| \geq C_b\sigma_b^n$.*

*Proof.* Given $\delta > 0$, define

$$I_1(\delta) := \mathbf{I}(b, \delta + \alpha) \quad \text{and} \quad I_{n+1}(\delta) := B_\alpha^+(I_n(\delta)), \;\; \forall\, n \geq 1,$$

inductively. Moreover, denote $b_n := Q_b^n(0)$ for every $n \in \mathbb{N}$ and define

$$p_\alpha(\delta) := \min\left\{n \geq 1 : |I_n(\delta)| \geq \frac{|b_n|}{5n^2}\right\} \quad \text{and} \quad N_\alpha := p_\alpha(4\alpha).$$

We claim that there exists a constant $K > 1$, depending only on $Q_b$ , such that

$$K^{-1} \cdot |I_{n+1}(\delta)| \leq |(Q_b^n)'(b)| \cdot \delta \leq |I_{n+1}(\delta)|, \quad \forall\, 0 < \alpha < \delta,\; 1 \leq n < p_\alpha(\delta). \tag{2.25}$$

By definition,

$$\theta := \sum_{n=1}^{p_\alpha(\delta)-1} \frac{|I_n(\delta)|}{|b_n|} < \frac{1}{5} + \sum_{n=1}^{\infty} \frac{1}{5n(n+1)} = \frac{2}{5} \Longrightarrow \frac{\theta}{1-\theta} < \frac{2}{3} < \log 2,$$

so according to (2.21), for any sequence $\{y_i \in I_i(\delta)\}_{i=m}^n$ with $m < n \leq p_\alpha(\delta)$, we have:

$$\frac{1}{2} \cdot |(Q_b^{n-m})'(b_m)| < \prod_{i=m}^{n-1} |Q_b'(y_i)| < 2 \cdot |(Q_b^{n-m})'(b_m)|. \tag{2.26}$$

According to (2.17), for each $n \in \mathbb{N}$, there exists $y_n \in I_n(\delta)$, such that for $D_n := |Q_b'(y_n)| \neq 0$, we have:

$$|I_{n+1}(\delta)| = I_1(\delta) \cdot \prod_{k=1}^{n} D_k + 2\alpha \cdot (1 + \sum_{k=2}^{n} \prod_{j=k}^{n} D_j) = 2 \prod_{i=1}^{n} D_i \cdot (\delta + \alpha + \alpha \sum_{k=1}^{n} \prod_{j=1}^{k} D_j^{-1}), \quad \forall\, n \in \mathbb{N}.$$

Due to (2.26) and the Collet-Eckmann condition on $Q_b$,

$$\sum_{k=1}^{n}\prod_{i=1}^{k} D_i^{-1} < 2\sum_{k=1}^{\infty}|(Q_b^k)'(b)|^{-1} := K_0 < \infty, \quad 1 \le n < p_\alpha(\delta).$$

Then for $K = 4(K_0 + 2)$, (2.25) follows from the formula of $|I_{n+1}(\delta)|$ above and (2.26) .

Now let us turn to the proof of the lemma by applying (2.25). First recall that by the definition of $p_\alpha(\delta)$, $\frac{|b_{p_\alpha(\delta)}|}{5(p_\alpha(\delta))^2} \le |I_{p_\alpha(\delta)}| < 2$. Combining this fact with $\lim_{n\to\infty}\frac{\log|b_n|}{n} = 0$, we have:

$$\lim_{p_\alpha(\delta)\to\infty}\frac{\log|I_{p_\alpha(\delta)}|}{p_\alpha(\delta)} = 0.$$

Since $Q_b$ is Collet-Eckmann, according to the limit above and (2.25) for $n = p_\alpha(\delta) - 1$, we have $\lim_{\substack{\delta\to 0^+\\ 0<\alpha<\delta}} p_\alpha(\delta) = \infty$, and consequently

$$\lim_{\substack{\delta\to 0^+\\ 0<\alpha<\delta}}\frac{\log(|(Q_b^{p_\alpha(\delta)-1})'(b)|\cdot\delta)}{p_\alpha(\delta)} = 0. \tag{2.27}$$

As a result, for $N_\alpha = p_\alpha(4\alpha)$, given $\eta \in (0, \frac{1}{2})$, when $\alpha$ is small, $|(Q_b^{N_\alpha-1})'(b)| \ge \alpha^{\eta-1}$ and $I_n(4\alpha) \cap \mathbb{I}_{\alpha^\eta} = \varnothing$ for $1 \le n < N_\alpha$. Besides, from $|(Q_b^n)'(b)| \le R_b^{n+p_b}$, $R_b < 2$ and $\eta < \frac{1}{2}$ we know $4^{N_\alpha} \ge \alpha^{-1}$, provided that $\alpha > 0$ is small. The by choosing $\sigma_b$ close to 1, the proof of the first assertion in the lemma is completed.

To prove the second assertion, let us introduce some constants first. Since $Q_b$ is Collet-Eckmann, by definition and (2.11), there exist $C_0 \in (0, 1)$ and $\sigma_0 \in (1, \lambda_b]$, such that

- $$|(Q_b^n)'(b)| \ge C_0\sigma_0^{3n}, \quad \forall\, n \in \mathbb{N}_0;$$

- for every $\delta > 0$ and every $n \in \mathbb{N}$,

$$y, Q_b(y), \cdots, Q_b^{n-1}(y) \notin \mathbb{I}_{\frac{\delta}{2}},\ Q_b^n(x) \in \mathbb{I}_{2\delta} \Longrightarrow |(Q_b^n)'(y)| \ge 2C_0\sigma_0^n.$$

For $C_0$ and $\sigma_0$ introduced above, according to (2.27), there exists $\delta_0 \in (0, 1)$, such that

$$0 < \alpha < \delta \leq \delta_0 \Longrightarrow |(Q_b^{p_\alpha(\delta)-1})'(b)| \geq C_0^{-1}\sigma_0^{p_\alpha(\delta)} \cdot \delta^{-\frac{1}{2}}.$$

Then for $\delta = y^2$, by (2.26),

$$|\partial_y f_{p_\alpha(y^2)}(x, y)| \geq C_0^{-1}\sigma_0^{p_\alpha(y^2)}, \quad \forall\ (x, y) \in \mathbf{I}_{\beta_a} \times (\mathbf{I}_{\sqrt{\delta_0}} \setminus \mathbf{I}_{\sqrt{\alpha}}).$$

For this $\delta_0$, from (2.9) we know that there exist $n_0 \in \mathbb{N}$ and $\sigma_b \in (1, \sigma_0]$, such that

$$y, Q_b(y), \cdots, Q_b^{n_0-1}(y) \notin \mathbb{I}_{\frac{\delta_0}{2}} \Longrightarrow |(Q_b^{n_0})'(y)| \geq 2\sigma_b^{n_0}.$$

Then by continuity, provided that $\alpha > 0$ is small, for $y_i = f_i(x_0, y_0)$, $i \in \mathbb{N}$, we have:

$$y_0, y_1, \cdots, y_{n_0-1} \notin \mathbb{I}_{\delta_0} \Longrightarrow |\partial_y f_{n_0}(x_0, y_0)| \geq \sigma_b^{n_0},$$

and

$$y_0, y_1, \cdots, y_{k-1} \notin \mathbb{I}_{\delta_0},\ y_k \in \mathbb{I}_{\delta_0},\ k \leq n_0 \implies |\partial_y f_k(x_0, y_0)| \geq C_0\sigma_b^k.$$

As a result, for every $n \in \mathbb{N}$,

$$y_0, y_1, \cdots, y_{n-1} \notin \mathbb{I}_{\delta_0},\ y_n \in \mathbb{I}_{\delta_0} \Longrightarrow |\partial_y f_n(x_0, y_0)| \geq C_0\sigma_b^n,$$

and for $C_1 := (2\delta_0\sigma_b^{-1})^{n_0} > 0$,

$$y_0, y_1, \cdots, y_{n-1} \notin \mathbb{I}_{\delta_0} \Longrightarrow |\partial_y f_n(x_0, y_0)| \geq C_1\sigma_b^n.$$

Now for $C_b := (C_0^2\sigma_b^{-1}) \wedge C_1$, we are ready to deduce the lower bound of $|\partial_y f_n(z_0)|$ for the orbit appearing in the second assertion of the lemma. If $y_0, y_1, \cdots, y_{n-1} \notin \mathbb{I}_{\delta_0}$, there is nothing to prove. Otherwise, let us define a sequence of times $t_1 < t_2 < \cdots$ inductively. Let $t_1 \in \mathbb{N}_0$ be the first time such that $y_{t_1} \in \mathbb{I}_{\delta_0}$. Once $t_k$ is defined, let $p_k := p_\alpha(y_{t_k}^2)$ and let $t_{k+1} \geq t_k + p_k$ be the minimal time $t$ such that $y_t \in \mathbb{I}_{\delta_0}$ if it is well defined. Finally, let $s$ be the maximal $k$ such that $t_k \leq n$ is well defined. Then

$$|\partial_y f_{t_1}(z_0)| \geq C_0 \sigma_b^{t_1},$$

and for $1 \leq i < s$,

$$|\partial_y f_{t_{i+1}-t_i}(z_{t_i})| = |\partial_y f_{p_i}(z_{t_i})| \cdot |\partial_y f_{t_{i+1}-t_i-p_i}(z_{t_i+p_i})| \geq \sigma_b^{t_{i+1}-t_i}.$$

It follows that

$$|\partial_y f_{t_s}(z_0)| \geq C_0 \sigma_b^{t_s},$$

which completes the proof if $n = t_s$. Otherwise, either $t_s < n < t_s + p_s$, so we have $|y_n| > \alpha^\eta$ and

$$|\partial_y f_{n-t_s}(z_{t_s})| \geq |y_{t_s}| \cdot |(Q_b^{n-t_s-1})'(b_1)| \geq \sqrt{\alpha} \cdot C_0 \sigma_b^{n-t_s-1} ;$$

or $t_s + p_s \leq n < t_{s+1}$, and hence

$$|\partial_y f_{n-t_s}(z_{t_s})| = |\partial_y f_{p_s}(z_{t_s})| \cdot |\partial_y f_{n-t_s-p_s}(z_{t_s+p_s})| \geq C_0^{-1} C_1 \sigma_b^{n-t_s}.$$

The conclusion follows from the last three displayed inequalities and the choice of $C_b$.

□

*Remark.* We will fix the choice of $\eta = \eta_b$ as below whenever we apply Lemma 2.3.1 in this thesis.

$$\eta_b := \frac{\log \sigma_b}{100 \log 2}. \tag{2.28}$$

Therefore, actually the assumption $\lim\limits_{n\to\infty} \frac{\log |Q_b^n(0)|}{n} = 0$ can be relaxed to $\limsup\limits_{n\to\infty} \frac{\log \frac{1}{|Q_b^n(0)|}}{n} \le \epsilon$ for some $\epsilon > 0$ small enough.

As a corollary of Lemma 2.3.1, we have the following statement, which is closely related to [AV02, Proposition 6.2] and will be used to prove Proposition 5.2.1. The argument in [AV02] also works here, but since the setting is not exactly the same here and since the argument there can be simplified a little, let us provided a detailed proof here. Recall the notation $\check{I}_c^k$ introduced in (2.6).

**Lemma 2.3.2.** *The following statement holds when $\alpha > 0$ is small. Let $I_0 \subset \mathbf{I}_{\beta_a}$ be an arbitrary non-degenerate interval and let $J_0 \subset \Lambda_b$ be an interval with $|J_0| \ge \alpha^{1-\frac{5}{4}\eta_b}$. Then there exists $n \in \mathbb{N}$, such that $F^n(I_0 \times J_0)$ contains a rectangle $\check{I}_a \times J$, where $|\check{I}_b^k \setminus J| \le \alpha^{0.9}$ for some $k \in \mathbb{N}_0$.*

*Proof.* According to the fact that the backward $Q_a$-orbit of 0 is dense in $\mathbf{I}_{\beta_a}$ and the last assertion in Lemma 2.1.3, we know that there exists $n_a \in \mathbb{N}$, determined by $Q_a$, such that for every interval $I \subset \mathbf{I}_{\beta_a}$ with $|I| \ge \hat{r}_a$, $Q_a^n(I) \supset \check{I}_a^j$ for some $j \in \mathbb{N}_0$ when $n \ge n_a$. Let $x_0$ be the midpoint of $I_0$. Then from the exponential shrinking property of $Q_a$ we know that when $n \ge n_a + \frac{\log \frac{2}{|I_0|}}{\log \lambda_a}$, $I := \mathrm{comp}(x_0, n - n_a, \hat{r}_a) \subset I_0$ and $Q_a^n(I) \supset \check{I}_a^j$ for some $j \in \mathbb{N}_0$. Moreover, from Lemma 2.2.2 we know that

$$|f_{n-n_a}(x,y) - f_{n-n_a}(x_0,y)| \le \hat{A}_0\alpha, \text{ and hence } |f_n(x,y) - f_n(x_0,y)| \le A\alpha, \ \forall\ (x,y) \in I \times J_0,$$

where $A > A_0$ is some constant determined by $(a,b)$. Also recall that $Q_a^{\hat{p}_a}(\check{I}_a) = \check{I}_a$. Thus to prove the lemma, by continuity, it suffices to show that for $J_m := f_m(\{x_0\} \times J_0)$, $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $|\check{I}_b^k \setminus J_n| \le \alpha^{0.95}$ for some $k \in \mathbb{N}_0$.

We start with showing that $|J_n| \geq \frac{1}{2}\alpha^{\eta_b}$ for some $n \in \mathbb{N}$. To see this, let $m \in \mathbb{N}_0$ be minimal, such that $J_m \cap \mathbf{I}_{\sqrt{\alpha}} \neq \emptyset$, whose existence is guaranteed by assertion (2) in Lemma 2.3.1. If $J_m \subset \mathbb{I}_{2\sqrt{\alpha}}$, then again by assertion (2) in Lemma 2.3.1, $|J_m| \geq C_b \sigma_b^m |J_0|$; otherwise, $|J_m| \geq \sqrt{\alpha}$. In both cases, there exists a subinterval $K$ of $J_m$, which satisfies that

$$K \subset \mathbb{I}_{2\sqrt{\alpha}} \setminus \{0\} \quad \text{and} \quad |K| \geq \min\{\frac{C_b}{2}|J_0|, \sqrt{\alpha}\}.$$

Then by assertion (1) in Lemma 2.3.1, when $\alpha > 0$ is small,

$$|J_{m+N_\alpha}| \geq \alpha^{\eta_b - 1} \int_K |y| \,\mathrm{d}\, y \geq \frac{1}{2}\alpha^{\eta_b - 1}|K|^2 \geq \min\{\alpha^{-\frac{\eta_b}{5}}|J_0|, \frac{1}{2}\alpha^{\eta_b}\}.$$

We can replace $J_0$ with $J_{m+N_\alpha}$ and apply this argument again. Since $|J_0| \geq \alpha^{1-\frac{5}{4}\eta_b}$, to end up with an interval of length no less than $\frac{1}{2}\alpha^{\eta_b}$, we only need to repeat this argument no more than $\lceil 5\eta_b^{-1} \rceil$ times. This verifies the statement at the beginning of this paragraph.

To complete the proof, firstly, from the conclusion in the last paragraph we know that without loss of generality, we may assume that $|J_0| \geq \frac{1}{2}\alpha^{\eta_b}$. Secondly, according to (2.20),

$$|J_k| > |Q_b^k(J_0)| - \frac{4^{k+1}}{3} \cdot \alpha, \quad \forall\, k \in \mathbb{N}_0.$$

Thirdly, as in the discussion about $Q_a$ at the beginning of the proof, we know that there exists $n_b \in \mathbb{N}$, determined by $Q_b$, such that for any interval $J \subset \Lambda_b$ with $|J| \geq \hat{r}_b$, we have $Q_b^{n_b}(J) \supset \check{I}_b^k$ for some $k \in \mathbb{N}_0$. On the one hand, combining this with the exponential shrinking property we know that for $n := n_b + \left\lfloor \frac{3\eta_b \log \alpha^{-1}}{2 \log \lambda_b} \right\rfloor$, $Q_b^n(J_0) \supset \check{I}_b^k$ for some $k \in \mathbb{N}_0$; on the other hand, due to the choice of $n$ and $\eta_b$, $n < \frac{\log \frac{1}{\alpha}}{50 \log 2}$, provided that $\alpha > 0$ is small, and therefore from the displayed inequality above we know $|J_n| > |Q_b^n(J_0)| - \alpha^{0.95}$. The proof is completed. $\square$

Chapter 3

# Iteration of horizontal curves

In this chapter, motivated by the ideas in [HS13, GS14], we use complex analysis techniques to study analytical properties of the image of a horizontal curve under $F$-iteration, and our main goal is to obtain Proposition 3.4.4.

## 3.1 A family of analytic functions

Given an interval $I \subset \mathbb{R}$ and a function $Y : I \to \mathbb{R}$, we always denote the associated graph function by $\hat{Y}$, i.e.

$$\hat{Y} : I \to I \times \mathbb{R}, \quad x \mapsto (x, Y(x)).$$

Given a subinterval $J$ of $I$, denote the restriction of $Y$ to $J$ by $Y|_J$. If $Q_a^n$ is injective on $J$ for some $n \in \mathbb{N}$, then let us use $F_*^n(Y|_J)$ to denote the "push-forward" of $Y|_J$ by $F^n$, i.e.

$$F_*^n(Y|_J) : Q_a^n(J) \to \mathbb{R}, \quad Q_a^n(x) \mapsto f_n(\hat{Y}(x)).$$

Also recall the notations $\mathrm{comp}(x, n, r)$, $\mathbb{C}_I$ and $\mathrm{Inv}(x, n)$ defined in § 1.4.

**Definition 3.1.1.** Fix $r > 0$. Given $x \in \mathbf{I}_{\beta_a}$ and $n \in \mathbb{N}$, $(x, n)$ is called an $r$-**admissible pair**, if $Q_a^n$ maps $\mathrm{comp}(x, n, 2r)$ to $\mathbb{I}(Q_a^n(x), 2r)$ bijectively. Given an $r$-admissible pair

$(x,n)$ and a horizontal curve $Y_0 \equiv y$ for some $y \in \Lambda_b$, the curve $F_*^n(Y_0|_{\mathrm{comp}(x,n,r)})$ is called an $r$-**admissible curve** centered at $Q_a^n(x)$.

By definition, if $(x,n)$ is an $r$-admissible pair, then $(Q_a^k(x), n-k)$ is also an $\tilde{r}$-admissible pair for $0 \le k < n$ and $\tilde{r} \in (0,r]$; moreover, $\mathrm{Inv}(x,n)$ extends to a univalent function on $\mathbb{C}_{\mathbb{I}(Q_a^n(x),2r)}$. As a consequence, an $r$-admissible curve centered at $x_* \in \mathbf{I}_{\beta_a}$ will be often viewed as a holomorphic function on $\mathbb{D}(x_*, 2r) \subset \mathbb{C}_{\mathbb{I}(x_*,2r)}$.

Following [HS13, GS14], we introduce a family of analytic functions as follows.

**Definition 3.1.2.** Given $r > 0$, let $\mathcal{T}_r$ be the collection of holomorphic functions $T$ satisfying the following assumptions:

- there exists an $r$-admissible pair $(x,n)$, such that $\mathrm{dom}(T) = \mathbb{C}_{\mathbb{I}(Q_a^n(x),2r)}$;
- $T$ has the form
$$T = (\varphi \circ \tau_1)' + \sum_{k=2}^{n} c_k (\varphi \circ \tau_k)', \tag{3.1}$$
where $c_k \in \mathbb{C}$, $\tau_k = \mathrm{Inv}(Q_a^{n-k}(x), k)$ for $1 \le k \le n$, $c_1 = 1$ and $|c_k| \le \hat{R}_b^{k-j+p_b} |c_j|$ when $1 \le j < k \le n$.

Moreover, define

$$\mathcal{S}_r := \{S : \mathbb{C}_{\mathbb{I}_{2r}} \to \mathbb{C} \mid S = \varphi' + cT, \text{ where } c \in \mathbf{I}_4 \text{ and } T \in \mathcal{T}_r \text{ with } \mathrm{dom}(T) = \mathbb{C}_{\mathbb{I}_{2r}}\}.$$

In below we will use compactness arguments for certain families of holomorphic functions, so let us recall the following two classical results in complex analysis.

**Lemma 3.1.1** (Montel's theorem)**.** *Let $U$ be an open set in the complex plane and let $\mathcal{H}$ be a locally uniformly bounded family of holomorphic functions defined on $U$. Then every infinite sequence in $\mathcal{H}$ contains a subsequence that is locally uniformly convergent to some holomorphic function on $U$.*

**Lemma 3.1.2** (Koebe distortion theorem)**.** *Let $h : \mathbb{D}_1 \to \mathbb{C}$ be a univalent holomorphic function with $h(0) = 0$ and $h'(0) = 1$. Then we have:*

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2} \quad \textit{and} \quad \frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad \forall\, z \in \mathbb{D}_1.$$

For the families $\mathcal{T}_r$ and $\mathcal{S}_r$ introduced in Definition 3.1.2, we have:

**Lemma 3.1.3.** *For every $r > 0$, there exists $\hat{\delta}_1(r) > 0$, such that for every $T \in \mathcal{T}_r$ with* $\mathrm{dom}(T) = \mathbb{C}_{\mathbb{I}(x,2r)}$, *we have:*

$$\|T\|_{\mathbb{D}(x,r)} \ge 2\hat{\delta}_1(r). \tag{3.2}$$

*In addition, for every $S \in \mathcal{S}_r$, we have:*

$$\sup_{z\in\mathbb{D}_r} |S(z) + S(-z)| \ge 2\hat{\delta}_1(r). \tag{3.3}$$

*Proof.* Let us focus on (3.2) first. Denote $V := \mathbb{C}_{\mathbb{I}_{2r}}$. By definition, if $T \in \mathcal{T}_r$ with $\mathrm{dom}(T) = \mathbb{C}_{\mathbb{I}(x,2r)}$, then $T$ has the form $T = \sum_{i=1}^{n} c_i \cdot (\varphi \circ \tau_i)'$, where $|c_i| \le \hat{R}_b^{i-1+p_b}$ and $\tau_i$ is univalent on $x + V$ for $1 \le i \le n$. Due to (2.12) and Koebe's distortion theorem, for any compact subset $K$ of $V$, there exists a constant $C > 0$, determined by $r$ and $K$, such that $\|\tau_i'\|_{x+K} \le C\lambda_a^{-i}$ for $1 \le i \le n$. Replacing $K$ with a larger compact subset of $V$ if necessary, we may assume that for every $z \in K$, the line segment $\{tz : t \in [0,1]\}$ is contained in $K$. Then for $\rho := \max\{|z| : z \in K\}$,

$$\tau_i(x+K) \subset \mathbf{D}(\tau_i(x), C\rho\lambda_a^{-i}) \subset \mathbf{D}_{C\rho+2}, \quad 1 \le i \le n.$$

Combining the known facts above and the basic assumption $\hat{R}_b < \lambda_a$ yields that

$$\|T\|_{x+K} \le \sum_{i=1}^{n} |c_i| \cdot \|\varphi'\|_{\tau_i(x+K)} \cdot \|\tau_i'\|_{x+K} \le C \cdot \|\varphi'\|_{\mathbf{D}_{C\rho+2}} \cdot \sum_{i=1}^{n} \hat{R}_b^{i-1+p_b} \lambda_a^{-i} \le \tilde{C},$$

where $\tilde{C} > 0$ depends only on $(a, b)$, $r$ and $K$. That is to say,

$$\mathcal{H} := \{h : V \to \mathbb{C} \mid h(z) = T(x+z), \text{ where } x \in \mathbf{I}_{\beta_a}, T \in \mathcal{T}_r \text{ and } \operatorname{dom}(T) = \mathbb{C}_{\mathbb{I}(x,2r)}\}$$

is a locally uniformly bounded family of holomorphic functions on $V$. Therefore, by Montel's theorem, to prove (3.2), it suffices to show that there exist $z_0 \in V$ and $\delta > 0$, such that for every $h \in \mathcal{H}$, $|h(z_0)| \geq \delta$.

To this end, given $h \in \mathcal{H}$ of the form $h(z) = \sum_{i=1}^{n} c_i(\varphi \circ \tau_i)'(x+z)$, let us estimate the lower bound of $|h(z)|$ for $z \in V$ with $|z|$ large(independent of $h$). To begin with, note that $|x| \leq 2$ and hence $|x+z| \asymp |z|$. From assertion (3) in Corollary 2.1.6 we know that there exists a unique $1 \leq k \leq n$, such that $|\tau_i(x+z)| \geq 3$ if and only if $1 \leq i \leq k$. Moreover, on the one hand, according to (2.4),

$$|(\varphi \circ \tau_i)'(x+z)| \asymp |\tau_i(x+z)|^{d_\varphi - 1} \cdot |\tau_i'(x+z)| \asymp 2^{-i}|z|^{d_\varphi \cdot 2^{-i} - 1}, \quad 1 \leq i \leq k;$$

on the other hand, according to (2.13),

$$|(\varphi \circ \tau_i)'(x+z)| \leq C_1|\tau_i'(x+z)| \leq C_2 \hat{r}_a^{-1} \cdot 2^{-k} \lambda_a^{k-i} |z|^{d_\varphi \cdot 2^{-k} - 1}, \quad k < i \leq n,$$

where $C_1, C_2 > 0$ depend only on $\varphi$. It follows that when $z \in V$ and $|z|$ is large,

$$|(\varphi \circ \tau_1)'(x+z)| \asymp |z|^{\frac{d_\varphi}{2} - 1} \text{ and } \left|\sum_{i=2}^{n} c_i(\varphi \circ \tau_i)'(x+z)\right| \leq C_3 \cdot |z|^{\frac{d_\varphi}{4} - 1} \Longrightarrow |h(z)| \geq C_4 \cdot |z|^{\frac{d_\varphi}{2} - 1},$$

where $C_3, C_4 > 0$ depend only on $(a, b)$ and $\varphi$. This proves the existence of desired $z_0$ and $\delta$ at the end of the last paragraph, and hence completes the proof of (3.2).

Using the additional assumption that $d_\varphi$ is odd, the proof of (3.3) is quite similar. Let

$$\mathcal{H}' := \{h : V \to \mathbb{C} \mid h(z) = S(z) + S(-z) \text{ for some } S \in \mathcal{S}_r\}.$$

A totally similar argument shows that $\mathcal{S}_r$, and hence $\mathcal{H}'$, are locally uniformly bounded families of holomorphic functions. Moreover, when $z \in V$ and $|z|$ is large, for any $S \in \mathcal{S}_r$ of the form $S = \varphi' + cT$, by the assumption that $d_\varphi$ is odd, we have

$$|\varphi'(z) + \varphi'(-z)| \asymp |z|^{d_\varphi - 1} \quad \text{and} \quad |T(z)| \asymp |z|^{\frac{d_\varphi}{2} - 1}.$$

It follows that for $h(z) = S(z)+S(-z)$, $|h| \asymp |z|^{d_\varphi - 1}$, which, together with a simple argument using Montel's theorem, implies (3.3). □

*Remark.* It should be noted that without assuming $\varphi$ of odd degree, (3.3) could fail in general. For example, given $|c| \leq \hat{R}_b$, let $\varphi(x) = Q_a(x) + cx$ and let $S = \varphi' + \sum_{i=1}^{n} (-c)^i (\varphi \circ \tau_i)'$ be in $\mathcal{S}_r$ for some $r > 0$. Then actually $S(x) = -(2x + (-c)^{n+1} \tau_n'(x))$, and hence $\sup_{z \in \mathbb{D}_r} |S(z) + S(-z)|$ is exponentially small in $n$.

## 3.2 Lower bound of oscillation

In this section, we take the first step for describing non-flatness of an arbitrary $r$-admissible curve by using $\alpha T$ to approximate its derivative with appropriate chosen $T \in \mathcal{T}_r$.

**Lemma 3.2.1.** *For every $r \in (0, \frac{\hat{r}_a}{2})$, the following statements hold when $\alpha > 0$ is sufficiently small. Let $Y$ be an $r$-admissible curve centered at $x_* \in \mathbf{I}_{\beta_a}$. Then we have:*

$$\|Y - Y(x_*)\|_{\mathbb{D}(x_*, 2r)} \leq \hat{A}_0 \alpha \quad \textit{and} \quad \hat{\delta}_1(r)\alpha \leq \|Y'\|_{\mathbb{D}(x_*, r)} \leq \hat{A}_0 r^{-1} \alpha. \tag{3.4}$$

*In addition, if $x_* = 0$, then for $Z(x) := f_1(\hat{Y}(x))$,*

$$\sup_{z \in \mathbb{D}_r} \|Z'(z) + Z'(-z)\| \geq \hat{\delta}_1(r)\alpha. \tag{3.5}$$

*Proof.* By definition, there exist $n \in \mathbb{N}$, $x_0 \in Q_a^{-n}(x_*)$ and $y_0 \in \Lambda_b$, such that for $\tau_n := \mathrm{Inv}(x_0, n)$, $Y(x) = f_n(\tau_n x, y_0)$. Denote $(x_k, y_k) := F^k(x_0, y_0)$, $\tau_k := \mathrm{Inv}(x_{n-k}, k)$ and $Y_k(x) := f_k(\tau_n x, y_0)$ for $1 \leq k \leq n$. In particular, $x_* = x_n$, $Y = Y_n$ and $Y(x_*) = y_n$. Since $2r < \hat{r}_a$, from Lemma 2.2.2 (taking $r = 0$ in (2.23)) we know that

$$\|Y_k - y_k\|_{\mathbb{D}(x_*,2r)} \leq \hat{A}_0 \lambda_a^{k-n} \alpha, \quad 1 \leq k \leq n, \tag{3.6}$$

and

$$\|\partial_y f_j \circ \hat{Y}_k\|_{\mathbb{D}(x_*,2r)} \leq \hat{R}_b^{j+p_b}, \quad 1 \leq j + k \leq n. \tag{3.7}$$

In particular, taking $k = n$ in (3.6), we have $\|Y_n - y_n\|_{\mathbb{D}(x_*,2r)} \leq \hat{A}_0 \alpha$. Then by Cauchy's integral formula,

$$\|Y_n'\|_{\mathbb{D}(x_*,r)} \leq \|Y_n - y_n\|_{\mathbb{D}(x_*,2r)} \cdot r^{-1} \leq \hat{A}_0 r^{-1} \alpha,$$

which proves the upper bound part of $\|Y_n'\|$ in (3.4).

To obtain the lower bound, the basic strategy is to approximate $Y'$ by $\alpha T$ for some $T \in \mathcal{T}_r$. From

$$Y_k' = (Q_b' \circ Y_{k-1}) \cdot Y_{k-1}' + \alpha\,(\varphi \circ \tau_{n-k+1})' \quad \text{and} \quad \partial_y f_k \circ \hat{Y}_{n-k} = \prod_{j=0}^{k-1} Q_b' \circ Y_{j+n-k}, \quad 1 \leq k \leq n,$$

we know that

$$Y' = \alpha \cdot (\varphi \circ \tau_1)' + \partial_y f_1 \circ \hat{Y}_{n-1} \cdot Y_{n-1}' = \cdots = \alpha \sum_{k=0}^{n-1} \left(\partial_y f_k \circ \hat{Y}_{n-k}\right) \cdot (\varphi \circ \tau_{k+1})'. \tag{3.8}$$

Denote $c_{k+1} := \partial_y f_k(\hat{Y}_{n-k}(x_*))$ for $0 \leq k \leq n-1$. By Definition 3.1.2, (3.7) and (3.8),

$$T := \sum_{k=1}^{n} c_k \cdot (\varphi \circ \tau_k)' \in \mathcal{T}_r \,, \quad \text{and} \quad Y'(x_*) = \alpha \cdot T(x_*).$$

It follows that

$$\|Y' - \alpha T\|_{\mathbb{D}(x_*,r)} \leq \alpha \sum_{k=0}^{n-1} \|\partial_y f_k \circ \hat{Y}_{n-k} - c_{k+1}\|_{\mathbb{D}(x_*,r)} \cdot \|(\varphi \circ \tau_{k+1})'\|_{\mathbb{D}(x_*,r)}. \tag{3.9}$$

Then to control the upper bound of $\|Y' - \alpha T\|_{\mathbb{D}(x_*,r)}$, we have to estimate the upper bounds of the two factors of each summand on the right hand side of (3.9). On the one hand, since $\tau_k$ is univalent on $\mathbb{D}(x_*, 2r)$ and since $2r < \hat{r}_a$, by (2.12),

$$\|\tau_k'\|_{\mathbb{D}(x_*,r)} \leq r^{-1}\lambda_a^{-k} \Longrightarrow \|(\varphi \circ \tau_k)'\|_{\mathbb{D}(x_*,r)} \leq C_1 r^{-1}\lambda_a^{-k}, \quad 1 \leq k \leq n, \tag{3.10}$$

where $C_1 := \|\varphi'\|_{\mathbb{D}_3}$. On the other hand, noting that

$$\partial_y f_k \circ \hat{Y}_{n-k} = (-2)^k \prod_{j=0}^{k-1} Y_{j+n-k},$$

we have

$$\partial_y f_k \circ \hat{Y}_{n-k} - c_{k+1} = -2 \sum_{j=0}^{k-1} \partial_y f_j(x_{n-k}, y_{n-k}) \cdot (Y_{j+n-k} - y_{j+n-k}) \cdot \left(\partial_y f_{k-j-1} \circ \hat{Y}_{j+n-k+1}\right).$$

Therefore, due to (3.6), (3.7) and the expression above, we have:

$$\|\partial_y f_k \circ \hat{Y}_{n-k} - c_{k+1}\|_{\mathbb{D}(x_*,r)} \leq 2\hat{A}_0 \hat{R}_b^{k+2p_b-1} \sum_{j=0}^{k-1} \lambda_a^{j-k} \cdot \alpha \leq C_2 \hat{R}_b^k \alpha,$$

where $C_2 := 2\hat{A}_0 \hat{R}_b^{2p_b-1}(\lambda_a - 1)^{-1}$. Combining (3.9), (3.10) and the inequality above yields:

$$\|Y' - \alpha T\|_{\mathbb{D}(x_*,r)} \leq C_1 C_2 r^{-1} \alpha^2 \sum_{k=0}^{n-1} \hat{R}_b^k \cdot \lambda_a^{-k-1} \leq C_3 r^{-1} \alpha^2, \tag{3.11}$$

where $C_3 := C_1C_2(\lambda_a - \hat{R}_b)^{-1}$. According to the inequality above and (3.2), $\|Y'\|_{\mathbb{D}(x_*,r)} \geq \hat{\delta}_1(r)\alpha$, provided that $\alpha > 0$ is small enough. This proves (3.4).

When $x_* = 0$, the proof of (3.5) is similar to and based on the proof of (3.4). Note that by definition,

$$Z' = \alpha \cdot \varphi' + (Q_b' \circ Y) \cdot Y'.$$

For the function $T$ introduced before, let $S := \varphi' + Q_b'(Y(x_*)) \cdot T \in \mathcal{S}_r$. Due to (3.4), (3.11) and the choice of $S$,

$$\|Z' - \alpha S\|_{\mathbb{D}_r} \leq 4 \cdot \|Y' - \alpha T\|_{\mathbb{D}_r} + 2 \cdot \|Y - Y(x_*)\|_{\mathbb{D}_r} \cdot \|Y'\|_{\mathbb{D}_r} \leq (4C_3 + 2\hat{A}_0^2)r^{-1}\alpha^2.$$

Then (3.5) follows from (3.3), provided that $\alpha > 0$ is small enough. $\square$

As an immediate corollary of Lemma 3.2.1, we have:

**Corollary 3.2.2.** *Given $r \in (0, \frac{\hat{r}_a}{2})$, there exists $\hat{\delta}_2(r) > 0$, such that when $\alpha > 0$ is small, for every $x_* \in \mathbf{I}_{\beta_a}$ and every $r$-admissible curve $Y$ centered at $x_*$, we have:*

$$\|Y - Y(x_*)\|_{\mathbb{I}(x_*,r)} \geq \hat{\delta}_2(r)\alpha. \tag{3.12}$$

*In addition, when $x_* = 0$, we have:*

$$\left|\{\, x \in \mathbb{I}_r : |f_1(\hat{Y}(x)) - f_1(\hat{Y}(-x))| \leq \hat{\delta}_2(r)\alpha \,\}\right| \leq r. \tag{3.13}$$

*Proof.* To prove (3.12), define

$$\mathcal{H}_r := \{h : \mathbb{D}_{2r} \to \mathbb{D}_{\hat{A}_0} \mid h \text{ is holomorphic, } h(0) = 0 \text{ and } \|h'\|_{\mathbb{D}_r} \geq \hat{\delta}_1(r)\}.$$

By definition, $0 \notin \mathcal{H}_r$, and by Montel's theorem, $\mathcal{H}_r$ is closed under locally uniform convergence. As a result, there exists $\hat{\delta}_2(r) > 0$, such that $\|h\|_{\mathbb{I}_r} \geq \hat{\delta}_2(r)$ for every $h \in \mathcal{H}_r$. By Lemma 3.2.1, given $x_* \in \mathbf{I}_{\beta_a}$ and an $r$-admissible curve $Y$ centered at $x_*$, if we define $h(z) := \alpha^{-1}(Y(x_* + z) - Y(x_*))$ on $\mathbb{D}_{2r}$, then $h \in \mathcal{H}_r$. (3.12) follows.

The proof of (3.13) is similar. Let $\mathcal{H}'$ be the collection of holomorphic functions $h$ defined on $\mathbb{D}_{2r}$ that satisfies the following properties:

- $h$ is an odd function and $h$ is real-valued on $\mathbb{I}_{2r}$;
- $\|h\|_{\mathbb{D}_{2r}} \leq 8\hat{A}_0 + 2$ and $\|h'\|_{\mathbb{D}_{2r}} \geq \hat{\delta}_1(r)$.

Then by definition and Montel's theorem, $\mathcal{H}'$ is closed under locally uniform convergence; moreover, $0 \notin \mathcal{H}'$. In particular, $\lim_{\delta \to 0^+} |\{x \in \mathbb{I}_r : |h(x)| \leq \delta\}| = 0$ for every $h \in \mathcal{H}'$ uniformly. By choosing $\hat{\delta}_2(r) > 0$ smaller if necessary, it follows that $|\{x \in \mathbb{I}_r : |h(x)| \leq \hat{\delta}_2(r)\}| \leq r$ for every $h \in \mathcal{H}'$. Now for $Y$ appearing in (3.13), denote $Z(x) := f_1(\hat{Y}(x))$. Then $Z$ extends to a holomorphic function on $\mathbb{D}_{2r}$. On the one hand, by Lemma 3.2.1,

$$\sup_{z \in \mathbb{D}_r} |Z'(z) + Z'(-z)| \geq \hat{\delta}_1(r) \cdot \alpha\,;$$

on the other hand, since

$$|Z(z) - Z(-z)| \leq |Y(z) + Y(-z)| \cdot (|Y(z) - Y(0)| + |Y(-z) - Y(0)|) + \alpha \cdot (|\varphi(z)| + |\varphi(-z)|),$$

$$\sup_{z \in \mathbb{D}_{2r}} |Z(z) - Z(-z)| \leq 8 \cdot \|Y - Y(0)\|_{\mathbb{D}_{2r}} + 2\alpha \leq (8\hat{A}_0 + 2)\alpha.$$

That is to say, for $h(z) := \alpha^{-1}(Z(z) - Z(-z))$ on $\mathbb{D}_{2r}$, $h \in \mathcal{H}'$, which proves (3.13). □

## 3.3 Upper bound of oscillation

Recall $\mathscr{C}_{BC}$ introduced in Definition 2.1.1. In this subsection, to prove Lemma 3.3.2, we will make use of (2.2).

Let us begin with some elementary discussion on $Q_c$ for general parameter $c \in \mathbb{C}$. Given $z \in \mathbb{C}$, $n \in \mathbb{N}$, $r > 0$, denote

$$\rho_n^+(z,r) := \sup\{|w-z| : w \in \mathrm{Comp}(z,n,r)\} \quad \text{and} \quad \rho_n^-(z,r) := \inf\{|w-z| : w \notin \mathrm{Comp}(z,n,r)\}.$$

By definition, $\rho_n^+(z,r)$ is the minimum radius of open disks centered at $z$ that contain $\mathrm{Comp}(z,n,r)$, and $\rho_n^-(z,r)$ is the maximum radius of open disks centered at $z$ that are contained in $\mathrm{Comp}(z,n,r)$; moreover, for $k = 1, \cdots, n-1$,

$$\rho_n^+(z,r) \le \rho_k^+\left(z, \rho_{n-k}^+(Q_c^k(z), r)\right) \quad \text{and} \quad \rho_n^-(z,r) \ge \rho_k^-\left(z, \rho_{n-k}^-(Q_c^k(z), r)\right).$$

**Lemma 3.3.1.** *Fix an arbitrary $c \in \mathbb{C}$. Given $z \in \mathbb{C}$, $n \in \mathbb{N}$ and $r > 0$, if among the sets* $\mathrm{Comp}(Q_c^k(z), n-k, r)$*, $k = 0, \cdots, n-1$, at most $m$ of them contain* 0*, then*

$$\rho_n^+(z, \tau r) < 900 \cdot \tau^{\frac{1}{2^m}} \rho_n^-(z,r), \quad \forall\, \tau \in (0, 900^{-2^m}).$$

*Proof.* Let us start with two special cases: $m = 0$ or $m = n = 1$.

When $m = 0$, i.e. $Q_c^n : \mathrm{Comp}(z,n,r) \to \mathbb{D}(Q_c^n(z), r)$ is injective, then by Koebe's distortion theorem,

$$\rho_n^-(z,r) \ge \frac{r}{4|(Q_c^n)'(z)|},$$

and for every $\tau \in (0,1)$,

$$\rho_n^+(z,\tau r) \le \frac{\tau \cdot r}{|(Q_c^n)'(z)| \cdot (1-\tau)^2} \le \frac{4\tau}{(1-\tau)^2} \cdot \rho_n^-(z,r).$$

When $n = 1$, $Q_c : \mathrm{Comp}(z,1,r) \to \mathbb{D}(Q_c(z),r)$ is injective if and only if $0 \notin \mathrm{Comp}(z,1,r)$, i.e. $|z|^2 \ge r$. Direct calculation shows that:

$$\rho_1^-(z,r) = \sqrt{|z|^2+r} - |z|, \quad \text{and hence when } |z|^2 < r,\ \rho_1^-(z,r) > (\sqrt{2}-1)\sqrt{r}\ ;$$

$$\rho_1^+(z,r) = \begin{cases} |z| - \sqrt{|z|^2-r} & \le & \sqrt{r} & , \quad \text{when } |z|^2 \ge r \\ |z| + \sqrt{|z|^2+r} & < & (\sqrt{2}+1)\sqrt{r} & , \quad \text{when } |z|^2 < r \end{cases}.$$

Therefore, when $Q_c : \mathrm{Comp}(z,1,r) \to \mathbb{D}(Q_c(z),r)$ is not injective,

$$\rho_1^+(z,\tau r) < (3+2\sqrt{2})\sqrt{\tau} \cdot \rho_1^-(z,r), \quad \forall\, \tau > 0.$$

Combine these two special cases above, we know that if $0 \in \mathrm{Comp}(z,n,r)$ and $Q_c^{n-1}$ is injective on $\mathrm{Comp}(Q_c(z),n-1,r)$, then

$$\frac{\rho_n^+(z,\tau r)}{\rho_n^-(z,r)} < \frac{2(3+2\sqrt{2})}{1-\tau} \cdot \sqrt{\tau}, \quad \forall\, \tau \in (0,1).$$

In general, let $1 \le t_1 < \cdots < t_m \le n$ be all the times $1 \le t \le n$ such that $0 \in \mathrm{Comp}(Q_c^{n-t}(z),t,r)$. Let $r_0 = r$ and $\tau_0 = \tau$, and suppose that $r_i$ and $\tau_i$ have been defined for some $i < m$ by induction. Then for $r_{i+1} := \rho^-_{t_{i+1}-t_i}(Q_c^{n-t_{i+1}}(z), r_i)$, when $\tau_i \in (0,1)$,

$$\tau_{i+1} := \frac{\rho^+_{t_{i+1}-t_i}(Q_c^{n-t_{i+1}}(z), \tau_i r_i)}{r_{i+1}} \le \frac{2(3+2\sqrt{2})}{1-\tau_i} \cdot \sqrt{\tau_i}, \quad 0 \le i < m.$$

Moreover, when $\tau_m \in (0,1)$,

$$\frac{\rho_n^+(z,\tau r)}{\rho_n^-(z,r)} \le \frac{\rho^+_{n-t_m}(z,\tau_m r_m)}{\rho^-_{n-t_m}(z,r_m)} \le \frac{4\tau_m}{(1-\tau_m)^2}.$$

To complete the proof, by induction, we may assume that $\tau_0, \cdots, \tau_{k-1} < \frac{1}{900}$ for some $k \le m$. Then $\tau_{i+1} < 12\sqrt{\tau_i}$ for $i = 0, \cdots, k-1$ and hence

$$\tau_k < 12 \cdot (\tau_{k-1})^{\frac{1}{2}} < 12^{1+\frac{1}{2}} \cdot (\tau_{k-2})^{\frac{1}{4}} < \cdots < 12^2 \cdot \tau^{\frac{1}{2^k}} < \begin{cases} \frac{1}{900} & , \quad k < m \\ \frac{1}{5} & , \quad k = m \end{cases}.$$

The conclusion follows. □

**Lemma 3.3.2.** *The following statement holds when $n \in \mathbb{N}$ is large. Given $x \in \mathbf{I}_{\beta_a}$, there exists $\rho > 0$, such that*

$$\mathbb{D}(x, 5\rho) \subset \mathrm{Comp}(x, n, n^{-\frac{1}{\log\log n}}) \quad \textit{and} \quad \mathbb{I}(x, \rho) \supset \mathrm{comp}(x, n, n^{-\frac{1}{(\log\log n)^{0.9}}}). \tag{3.14}$$

*Proof.* Denote $a_t := Q_a^t(0)$ for every $t \in \mathbb{N}$. Denote $\mathrm{Comp}(x, n, n^{-\frac{1}{\log\log n}})$ by $U$ and let $1 \le t_1 < \cdots < t_m \le n$ be all the times from 1 to $n$ such that $0 \in Q_a^{n-t_i}(U)$. We claim that when $n$ is large(depending only on $Q_a$),

$$2^m < \frac{4\log n}{\log\log n}. \tag{3.15}$$

Once (3.15) is proved, we apply Lemma 3.3.1 to $Q_a^n$ on $U$ with $r = n^{-\frac{1}{\log\log n}}$ and $\tau r = n^{-\frac{1}{(\log\log n)^{0.9}}}$. Then when $n$ is large enough, $\tau < n^{-\frac{1}{(\log\log n)^{0.95}}} < 900^{-2^m}$ and hence

$$\frac{\rho_n^+(x, \tau r)}{\rho_n^-(x, r)} < 900 \cdot \tau^{\frac{1}{2^m}} < 900 \cdot e^{-\frac{(\log\log n)^{0.05}}{4}} < \frac{1}{5}.$$

By choosing $\rho = \rho_n^+(x, \tau r)$, the conclusion follows.

It remains to prove (3.15). To begin with, note that by definition, $0, a_{t_{i+1}-t_i} \in Q_a^{n-t_i}(U)$ for $i = 1, \cdots, m-1$, so by assertion (4) in Corollary 2.1.6,

$$|a_{t_{i+1}-t_i}| < \mathrm{diam}\,(Q_a^{n-t_i}(U)) \le \lambda_a^{-t_i+1} \cdot \left(2\hat{r}_a^{-1} n^{-\frac{1}{\log\log n}}\right)^{\frac{\log\lambda_a}{\log 6}} \le \lambda_a^{-t_i} \cdot n^{-\frac{1}{(\log\log n)^2}}, \quad i = 1, \cdots, m-1,$$

where the last inequality holds when $n$ is large enough.

By the Benedicks-Carleson recurrence assumption (2.2) on the critical orbit of $Q_a$, there exist $C \geq 1$ and $t_* \in \mathbb{N}$, such that $|a_t| \geq e^{-\sqrt{Ct}}$ when $t \geq t_*$. Then from the displayed inequality above we know that when $n$ is large, $t_{i+1} - t_i \geq t_*$ for $i = 1, \cdots, m-1$, and consequently

$$e^{-\sqrt{Ct_2}} < n^{-\frac{1}{(\log\log n)^2}} \Longrightarrow \tilde{C}t_2 > \log n$$

and

$$e^{-\sqrt{Ct_{i+1}}} < |a_{t_{i+1}-t_i}| < \lambda_a^{-t_i} \Longrightarrow t_{i+1} > \tilde{C}t_i^2, \quad i = 2, \cdots, m-1,$$

where $\tilde{C} = C^{-1}(\log\lambda_a)^2 \in (0,1)$. It follows that

$$n \geq t_m > \tilde{C} \cdot t_{m-1}^2 > \cdots > \tilde{C}^{2^k-1} t_{m-k}^{2^k} > \cdots > (\tilde{C}t_2)^{2^{m-2}} \Longrightarrow 2^m < \frac{4\log n}{\log\log n},$$

which completes the proof of (3.15) and the lemma.

□

*Remark.* The statement of Lemma 3.3.2 is a little flexible: it still holds when we replace the function $\kappa(n) = (\log\log n)^{0.9}$ in the expression $\mathrm{comp}(x, n, n^{-\frac{1}{\kappa(n)}})$ appearing in (3.14) by any function $\kappa : \mathbb{N} \to (0, +\infty)$ satisfying $\lim_{n\to\infty} \frac{\kappa(n)}{\log\log n} = 0$. This can be directly verified by checking the proof of Lemma 3.3.2. As a consequence, for results based on Lemma 3.3.2, namely Lemma 3.3.3, Proposition 3.4.4, Lemma 4.3.2, and finally Proposition 4.3.1, similar expressions $(\log\log n)^p$ for $p \in (0,1)$ can be replaced by such a function $\kappa$.

Now we use Lemma 3.3.2 to control upper bound of the oscillation for iteration of an admissible curve.

**Lemma 3.3.3.** *For every $r > 0$, there exists $n_a(r) \in \mathbb{N}$, such that when $\alpha > 0$ is small, for every $n \geq n_a(r)$, the following statement holds. Let $Y_0$ be an $r$-admissible curve centered at $x_* \in \mathbf{I}_{\beta_a}$. Then for every $x_0 \in \mathbb{I}(x_*, r)$, there exists $\rho \in (0, \frac{r}{5}]$, such that*

$\mathbb{I}(x_0,\rho) \supset \mathrm{comp}(x_0,n,\hat{r}_a)$ *and for* $Y_n(x) := f_n(\hat{Y}_0(x))$,

$$\|Y_n - Y_n(x_0)\|_{\mathbb{D}(x_0,5\rho)} < e^{n^{\frac{1}{(\log\log n)^{0.8}}}}\alpha. \tag{3.16}$$

*Proof.* Denote $(x_k,y_k) := F^k(\hat{Y}(x_0))$ and $Y_k(x) := f_k(\hat{Y}_0(x))$ for $k = 0,\cdots,n$. Given $r > 0$, when $n \in \mathbb{N}$ is large, for $n_2 = \left\lfloor \frac{\log n}{(\log\log n)^{0.85}} \right\rfloor$ and $n_1 := n - n_2$, $\lambda_a^{n_1} \geq r^{-1}$, $\lambda_a^{n_2} \geq n_1^{\frac{1}{(\log\log n_1)^{0.9}}}$ and Lemma 3.3.2 holds for $n_1$. Then there exists $\rho > 0$, such that

$$\mathbb{D}(x_0,5\rho) \subset \mathrm{Comp}(x_0,n_1,\hat{r}_a), \text{ and hence } 5\rho \leq \lambda_a^{-n_1} \leq r,$$

and

$$\mathbb{I}(x_0,\rho) \supset \mathrm{comp}(x_0,n_1,n_1^{-\frac{1}{(\log\log n_1)^{0.9}}}) \supset \mathrm{comp}(x_0,n_1,\lambda_a^{-n_2}) \supset \mathrm{comp}(x_0,n,\hat{r}_a).$$

It remains to prove (3.16). By the definition of $r$-admissible curve, there exist $m \in \mathbb{N}$, $x_{-m} \in Q_a^{-m}(x_0)$ and $y_{-m} \in \Lambda_b$, such that for $\tau_m := \mathrm{Inv}(x_{-m},m)$, $Y_0(x) = f_m(\tau_m x, y_{-m})$. Note that $\tau_m$ is well defined on $\mathbb{D}(x_0,5\rho)$ because of $5\rho \leq r$. Then according to (2.23), since $Q_a^{n_1}(\mathbb{D}(x_0,5\rho)) \subset \mathbb{D}(x_{n_1},\hat{r}_a)$,

$$\tau_m(\mathbb{D}(x_0,5\rho)) \subset \mathrm{Comp}(x_{-m},m+n_1,\hat{r}_a), \text{ so } Y_k(\mathbb{D}(x_0,5\rho)) \subset \mathbb{D}(y_k,\hat{A}_0\lambda_a^{k-n_1}\alpha), \ 0 \leq k \leq n_1.$$

In particular, for $k = n_1$, $\hat{Y}_{n_1}(\mathbb{D}(x_0,5\rho)) \subset \mathbb{D}(x_{n_1},\hat{r}_a) \times \mathbb{D}(y_{n_1},\hat{A}_0\alpha) \subset \mathbb{D}_3 \times \mathbb{D}_3$, so from Fact 2.2.3 we know that

$$\|Y_n - y_n\|_{\mathbb{D}(x_0,5\rho)} < 2^{d_\varphi \cdot 2^{n_2+2}}\hat{A}_0 \cdot \alpha.$$

Due to the choice of $n_2$, by assuming that $n$ is large enough, (3.16) follows from the inequality above. □

## 3.4 Transversality to horizontal lines

In this section, we prove Proposition 3.4.4 based on lower bound and upper bound obtained in the preceding sections.

**Lemma 3.4.1** ([BST03, Lemma 5.3])**.** *Given $l \in \mathbb{N}$ and an interval $I$, let $h : I \to \mathbb{R}$ be a $C^l$ function. If $|h^{(l)}| \geq \delta$ on $I$ for some $\delta > 0$, then for every $\epsilon > 0$ and every $y \in \mathbb{R}$,*

$$|\{\, x \in I : |h(x)| \leq \epsilon \,\}| \leq 2^{l+1} \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{l}}.$$

**Lemma 3.4.2.** *Let $h : \mathbb{I}_1 \to \mathbb{R}$ be a real analytic function, and suppose that for $M \geq 8\delta > 0$, $h$ extends to a holomorphic function $h : \mathbb{D}_5 \to \mathbb{D}(h(0), M)$ with $\|h - h(0)\|_{\mathbb{D}_1} \geq \delta$. Then for every $\epsilon \in (0, \frac{\delta}{100})$, we have*

$$|\{\, x \in \mathbb{I}_1 : |h(x)| \leq \epsilon \,\}| \leq \left(\frac{10M}{\delta}\right)^3 \cdot \left(\frac{\epsilon}{\delta}\right)^{\left(4 \log \frac{M}{\delta}\right)^{-1}}. \tag{3.17}$$

*Proof.* Let us do some estimates on $|h^{(l)}|$ first and then apply Lemma 3.4.1 to $h$. Firstly, note that

$$h(z) - h(0) = z \int_0^1 h'(tz)\,\mathrm{d}\,t,\ \forall\, z \in \mathbb{D}_1 \Longrightarrow \|h'\|_{\mathbb{D}_1} \geq \|h - h(0)\|_{\mathbb{D}_1} \geq \delta.$$

Secondly, by Cauchy's integral formula,

$$h^{(l)}(z) = \frac{l!}{2\pi\sqrt{-1}} \int_{|\zeta - z| = 4} \frac{(h(\zeta) - h(0))\,\mathrm{d}\,\zeta}{(\zeta - z)^{l+1}} \Longrightarrow |h^{(l)}(z)| \leq 4^{-l} l! M,\ \forall\, z \in \mathbb{D}_1,\ l \in \mathbb{N}. \tag{3.18}$$

Let $l_*$ be the minimal integer such that $\frac{l_*+2}{2^{l_*-1}} = \sum_{l=l_*}^{\infty} \frac{l+1}{2^l} \leq \frac{2\delta}{M}$. Then $\frac{2^{l_*-1}}{l_*+1} < \frac{M}{\delta}$ and from $M \geq 8\delta$ we know that $l_* \geq 6$. Fix $w \in \mathbf{D}_1$ with $|h'(w)| \geq \delta$. Given $z \in \mathbb{D}_1$, using the Taylor expansion of $h'$ at $z$ to evaluate $h'(w)$ and applying (3.18) to $l \geq l_*$, we obtain that

$$\delta \leq |h'(w)| \leq \sum_{l=0}^{\infty} \frac{|h^{(l+1)}(z)|}{l!}|w-z|^l \leq \sum_{l=0}^{l_*-1} \frac{|h^{(l+1)}(z)|}{l!} \cdot 2^l + \frac{M}{4} \cdot \sum_{l=l_*}^{\infty} \frac{l+1}{2^l},$$

Due to the choice of $l_*$, it follows that

$$\sum_{l=0}^{l_*-1} \frac{2^l}{l!}|h^{(l+1)}(z)| \geq \frac{\delta}{2}, \quad \forall\, z \in \mathbb{D}_1. \tag{3.19}$$

Now let us prove (3.17). Denote $m := \lceil l_*(l_*+1)M\delta^{-1} \rceil$. Recall that $l_* \geq 6$ and $\frac{2^{l_*-1}}{l_*+1} < \frac{M}{\delta}$. Then simple calculation shows that

$$l_* < 4\log\frac{M}{\delta} \quad \text{and} \quad 2^{l_*+1} \cdot 100^{\frac{1}{l_*}} \cdot m \cdot \frac{\delta}{M} < 5 \cdot 2^{l_*} \cdot l_*(l_*+1) < 10^3 \cdot \left(\frac{2^{l_*-1}}{l_*+1}\right)^2 < 10^3 \cdot \left(\frac{M}{\delta}\right)^2,$$

so it suffices to prove that

$$|\{\, x \in \mathbb{I}_1 : |h(x)| \leq \epsilon \,\}| \leq 2^{l_*+1} m \cdot \left(\frac{100\epsilon}{\delta}\right)^{\frac{1}{l_*}}.$$

As a direct corollary of (3.19), given $x \in \mathbb{I}_1$, there exists $1 \leq l_x \leq l_*$ such that

$$|h^{(l_x)}(x)| \geq 4^{-l_x}(l_x-1)! \cdot \delta.$$

Combining the inequality above and (3.18) for $l = l_x + 1$, due to the choice of $m$, we know that when $x' \in \mathbb{I}_1 \cap \mathbf{I}(x, m^{-1})$,

$$|h^{(l_x)}(x')| \geq |h^{(l_x)}(x)| - 4^{-l_x-1}(l_x+1)! \cdot Mm^{-1} \geq 4^{-l_x}(l_x-1)! \cdot (1 - \frac{l_x(l_x+1)M}{4m\delta}) \cdot \delta.$$

Since $\inf_{l\in\mathbb{N}} \frac{(l-1)!}{4^l} = \frac{3!}{4^4} = \frac{3}{128}$, from the choice of $m$ we know that

$$|h^{(l_x)}(x')| > \frac{\delta}{100}, \quad \forall x' \in \mathbb{I}_1 \cap \mathbf{I}(x, m^{-1}).$$

As a result, we can write $\mathbb{I}_1$ as a union of $m$ subintervals $J_1, \cdots, J_m$, such that for each $1 \le j \le m$, $|J_j| = 2m^{-1}$ and $|h^{(l_j)}| > \frac{\delta}{100}$ on $J_j$ for some $1 \le l_j \le l_*$. Then by Lemma 3.4.1, for every $1 \le j \le m$,

$$|\{x \in J_j : |h(x)| \le \epsilon\}| \le 2^{l_j+1}\left(\frac{100\epsilon}{\delta}\right)^{\frac{1}{l_j}} \le 2^{l_*+1}\left(\frac{100\epsilon}{\delta}\right)^{\frac{1}{l_*}},$$

which completes the proof. □

**Fact 3.4.3.** *Given $c \in \mathbb{R}$, for any interval $I \subset \mathbb{R}$ and any $n \in \mathbb{N}$, there exists an open subinterval $J$ of $I$, such that $Q_c^n$ maps $J$ to $\operatorname{int} Q_c^n(I)$ bijectively.*

*Proof.* When $n = 1$, if $0 \in \operatorname{int} I$, let $J$ be the connected component of $\operatorname{int} I \setminus \{0\}$ of larger length; if $0 \notin \operatorname{int} I$, then let $J = \operatorname{int} I$. When $n \ge 2$, on the one hand, apply the $n = 1$ case to $Q_c^{n-1}(I)$, we know that there exists an open subinterval $K$ of $Q_c^{n-1}(I)$, such that $Q_c$ maps $K$ to $\operatorname{int} Q_c^n(I)$ bijectively; on the other hand, by induction, there exists an open subinterval $L$ of $I$, such that $Q_c^{n-1}$ maps $L$ to $\operatorname{int} Q_c^{n-1}(I)$ bijectively. Since $K \subset \operatorname{int} Q_c^{n-1}(I)$, the conclusion holds for $J := L \cap Q_c^{-(n-1)}(K)$. □

**Proposition 3.4.4.** *For every $r > 0$, there exist $\hat{n}_a(r) \in \mathbb{N}$ and $\epsilon_r > 0$, such that when $\alpha > 0$ is small, the following statement holds. Given $x_* \in \mathbf{I}_{\beta_a}$, Let $Y_0$ be an $r$-admissible curve centered at $x_* \in \mathbf{I}_{\beta_a}$. Let $I$ be a subinterval of $\mathbb{I}(x_*, r)$ such that $|Q_a^n(I)| < \hat{r}_a$ for some $n \in \mathbb{N}$. Then for every $\epsilon \in (0, \epsilon_r)$,*

$$|\{x \in I : |f_n(\hat{Y}_0(x))| \le \alpha\epsilon\}| \le \begin{cases} \epsilon^{\frac{1}{\hat{n}_a(r)}} & n \le \hat{n}_a(r) \\ \epsilon^{n^{-\frac{1}{(\log\log n)^{0.7}}}} & n > \hat{n}_a(r) \end{cases}.$$

*Proof.* Denote the midpoint of $I$ by $x_0$ and denote $(x_k, y_k) := F^k(\hat{Y}_0(x_0))$ and $Y_k(x) = f_k(\hat{Y}_0(x))$ for $0 \le k \le n$. By definition, $Y_n$ extends to a holomorphic function on $\mathbb{D}_{\mathbb{I}(x_*,2r)}$. The basic idea is to show that there exists $\rho > 0$ with $\mathbb{I}(x_0, \rho) \supset I$ and $\mathbb{D}(x_0, 5\rho) \subset \mathbb{D}(x_*, 2r)$, such that for $h(z) := Y_n(x_0 + \rho z)$, $\|h - h(0)\|_{\mathbb{D}_5}$ has appropriate upper bound and $\|h - h(0)\|_{\mathbb{D}_1}$ has appropriate lower bound, so we can apply Lemma 3.4.2 to $h$. It will give a desired upper bound of

$$|\{x \in I : |Y_n(x)| \le \alpha\epsilon\}| \le \rho \cdot |\{x \in \mathbb{I}_1 : |h(x)| \le \alpha\epsilon\}|.$$

For the $n_a(r)$ appearing in Lemma 3.3.3, if $n \ge n_a(r)$, then from Lemma 3.3.3 we know that there exists $\rho \in (0, \frac{r}{5}]$, such that

$$\|Y_n - y_n\|_{\mathbb{D}(x_0,5\rho)} \le e^{n^{\frac{1}{(\log\log n)^{0.8}}}} \alpha \qquad \text{and} \qquad \mathbb{I}(x_0, \rho) \supset \mathrm{comp}(x_0, n, \hat{r}_a).$$

In particular, $|Q_a^n(\mathbb{I}(x_0, \rho))| \ge \hat{r}_a$. Then on the one hand, from $|Q_a^n(I)| < \hat{r}_a$ and $x_0 \in I$ we know that $I \subset \mathbb{I}(x_0, \rho)$. On the other hand, according to Fact 3.4.3, there exists an open interval $J \subset \mathbb{I}(x_0, \rho)$, such that $Q_a^n$ is injective on $J$ and $|Q_a^n(J)| \ge \hat{r}_a$. That is to say, $Z := F_*^n(Y_0|_J)$ contains a $\frac{\hat{r}_a}{4}$-admissible curve, and therefore by Corollary 3.2.2,

$$2 \cdot \|Y_n - y_n\|_{\mathbb{D}(x_0,\rho)} \ge \sup_{x,x' \in Q_a^n(J)} |Z(x) - Z(x')| \ge \hat{\delta}_2\Big(\frac{\hat{r}_a}{4}\Big) \cdot \alpha.$$

Then we can apply Lemma 3.4.2 to $h(z) = Y_n(x_0 + \rho z)$ with $M = e^{n^{\frac{1}{(\log\log n)^{0.8}}}} \alpha$ and $\delta = \frac{1}{2}\,\hat{\delta}_2\Big(\frac{\hat{r}_a}{4}\Big) \cdot \alpha$ to conclude that for some $N_0 \in \mathbb{N}$ and some $\epsilon_0 > 0$, both independent of $r$, such that when $n \ge N_0 \vee n_a(r)$ and $\epsilon \in (0, \epsilon_0)$,

$$|\{x \in \mathbb{I}_1 : |h(x)| \le \alpha\epsilon\}| \le \epsilon^{n^{-\frac{1}{(\log\log n)^{0.7}}}}.$$

If $n \le N_0 \vee n_a(r)$, without loss of generality, we may suppose that $|I| \le \frac{2}{5} r$, because otherwise, we can divide $I$ into 5 subintervals of equal lengths. Then we may simply choose $\rho = r/5$, so that $I \subset \mathbb{I}(x_0, \rho)$ and $\mathbb{D}(x_0, 5\rho) \subset \mathbb{D}(x_*, 2r)$. On the one hand, from Lemma 3.2.1 and Fact 2.2.3 we know that

$$\|Y_0 - y_0\|_{\mathbb{D}(x_*,2r)} \leq \hat{A}_0\alpha \Longrightarrow \|h - h(0)\|_{\mathbb{D}_5} \leq \|Y_n - y_n\|_{\mathbb{D}(x_*,2r)} < \hat{A}_0 \cdot 2^{d_\varphi \cdot 2^{n+2}}\alpha.$$

On the other hand, because $n \leq N_0 \vee n_a(r)$, there exists $\hat{r} \in (0, \frac{\hat{r}_a}{2})$, depending only on $a$ and $r$, such that $|Q_a^n(\mathbb{I}(x_0, r))| \geq 4\hat{r}$. By Fact 3.4.3, there exists an open subinterval $J$ of $\mathbb{I}(x_0, r)$, such that $Q_a^n$ maps $J$ to $\operatorname{int} Q_a^n(\mathbb{I}(x_0, r))$ bijectively. Then $F_*^n(\hat{Y}_0|_J)$ contains an $\hat{r}$-admissible curve. As the argument in the preceding paragraph, from Corollary 3.2.2 we know that $\|h - h(0)\|_{\mathbb{D}_1} \geq \frac{1}{2}\hat{\delta}_2(\hat{r})\alpha$. Applying Lemma 3.4.2 to $h$ with $M = \hat{A}_0 \cdot 2^{d_\varphi \cdot 2^{n+2}}\alpha$ and $\delta = \frac{1}{2}\hat{\delta}_2(\hat{r})\alpha$, the conclusion follows from Lemma 3.4.2 by choosing $\epsilon_r$ small and $\hat{n}_a(r) \geq N_0 \vee n_a(r)$ large.

□

# Chapter 4

# Slow recurrence to the critical line

This chapter aims at deducing the slow recurrence condition in the vertical direction with stretched exponential tail estimate, i.e. Proposition 4.3.1.

## 4.1 Induced Markov map of $Q_a$

Given $\rho \in (0, \beta_a)$, a set $\mathcal{Q}$ and a function $s : \mathcal{Q} \to \mathbb{N}$, we say that $(\rho, \mathcal{Q}, s)$ defines a **full induced Markov map** $G$ of $Q_a$ on $\mathbf{I}_{\beta_a}$ with range $\mathbb{I}_\rho$, if

- $\mathbb{I}_\rho$ is a **nice interval** of $Q_a$, i.e. $Q_a^n(\mathbb{I}_\rho) \cap \partial\mathbb{I}_\rho = \emptyset$ for every $n \in \mathbb{N}$;
- $\mathcal{Q}$ consists of pairwise disjoint open subintervals of $\mathbf{I}_{\beta_a}$ and $|\mathbf{I}_{\beta_a} \setminus \cup_{\omega \in \mathcal{Q}} \omega| = 0$;
- $\mathrm{dom}(G) = \cup_{\omega \in \mathcal{Q}} \omega$ and for every $\omega \in \mathcal{Q}$, $G = Q_a^{s(\omega)}$ on $\omega$ and $G(\omega) = \mathbb{I}_\rho$.

In the definition above, $\mathcal{Q}$ is called a **Markov partition** of $Q_a$ and $s$ is called the associated **inducing time** of $Q_a$. Moreover, we say that $G$ admits **$\tau$-scaled Koebe space** for some $\tau > 0$, if for every $\omega \in \mathcal{Q}$, $Q_a^{s(\omega)}$ maps some open neighborhood of $\omega$ to $\mathbb{I}_{(1+2\tau)\rho}$ bijectively.

It is well known that when $Q_a$ is Collet-Eckmann, there exist $\rho_n \to 0$ and $\tau_n \to \infty$, such that for every $n \in \mathbb{N}$, there exist Markov partition $\mathcal{Q}_n$ and associated inducing time function $s_n$, such that $(\rho_n, \mathcal{Q}_n, s_n)$ defines a full induced Markov map $G_n$ with $\tau_n$-scaled

Koebe space, and $|G_n'| \geq \tau_n$ on $\mathrm{dom}(G_n)$. Moreover, the tail of the inducing time function $\sum_{\substack{\omega\in Q_n \\ s_n(\omega)\geq s}} |\omega|$ can be chosen exponentially small in $s$. For the construction of Markov maps with arbitrary large scaled Koebe space, see, for example, [RLS]; for the exponential tail of inducing time, see, for example, [BLvS03].

In particular, for our purpose, we will fix a triple $(\rho_a, Q_a, s_a)$ that fulfills the following requirements.

- It defines a full induced Markov map $G_a$ of $Q_a$ that admits $\frac{1}{2}$-scaled Koebe space.
- For every $\omega \in Q_a$, either $\omega \subset \mathbb{I}_{\rho_a}$ or $\omega \cap \mathbb{I}_{\rho_a} = \emptyset$, and $|Q_a^n(\omega)| \leq \hat{r}_a$ when $0 \leq n \leq s_a(\omega)$; if $\omega \subset \mathbb{I}_{\rho_a}$ additionally, then $Q_a^{s_a(\omega)}(\mathbb{I}_{2\rho_a}) \supset \mathbb{I}_{2\rho_a}$;
- There exist $\gamma_a > 0$ and $\hat{C}_a \geq 1$, such that

$$\sum_{\substack{\omega\in Q_a \\ s_a(\omega)\geq s}} |\omega| \leq \hat{C}_a e^{-\gamma_a s}, \quad \forall\, s \in \mathbb{N}. \tag{4.1}$$

For the Markov map $G_a$ given above, let us introduce some more notations and conventions as follows.

Firstly, for every $n \geq 1$, let $Q_a^n := \bigvee_{i=0}^{n-1} G_a^{-i} Q_a$, or equivalently, $Q_a^n$ is the collection of connected components of $\mathrm{dom}(G_a^n)$. Note that by our assumption on $G_a$, for any $n \in \mathbb{N}$ and any $\omega \in Q_a^n$, when $1 \leq k \leq n$, $G_a^k|_\omega$ admits $\frac{1}{2}$-scaled Koebe space, so by Koebe's principle, the distortion of $G_a^k$ on $\omega$ is bounded by 9. As a result, for every measurable set $E \subset \mathbb{R}$,

$$\frac{1}{9} \cdot \frac{|G_a^k(\omega \cap E)|}{|G_a^k(\omega)|} \leq \frac{|E \cap \omega|}{|\omega|} \leq 9 \cdot \frac{|G_a^k(\omega \cap E)|}{|G_a^k(\omega)|}, \quad \forall\, \omega \in Q_a^n,\ 1 \leq k \leq n. \tag{4.2}$$

Secondly, we say that some $\mathbb{N} \cup \{\infty\}$-valued function $t$ defined on $\mathbf{I}_{\beta_a}$ is **compatible with** $G_a$, if $\mathrm{dom}(t) := \{x \in \mathbf{I}_{\beta_a} : t(x) < \infty\}$ has full measure in $\mathbf{I}_{\beta_a}$, the connected components of $\mathrm{dom}(t)$ are elements in $\cup_{n=1}^{\infty} Q_a^n$, and $t$ is a continuous(i.e. locally constant) function on dom($t$). For convenience, given a connected component $\omega$ of dom($t$), a subinterval $J$ of $\omega$ and $x \in \omega$, let us denote $t(\omega) := t(J) := t(x)$. As a first example, we consider $s_a$ as $\mathbb{N} \cup \{\infty\}$-valued function compatible with $G_a$ in the natural way. Functions $s_a^n$, $\iota_n$, $\zeta_n$ and $\xi_n$ introduced below are also such kind of functions.

Thirdly, given $n \in \mathbb{N}$, define $s_a^n(x) := \sum_{k=0}^{n-1} s_a(G_a^k x)$ for any $x \in \mathbf{I}_{\beta_a}$, i.e. $s_a^n$ is the inducing time of $G_a^n$, so by definition, $\mathrm{dom}(s_a^n) = \mathrm{dom}(G_a^n)$. Note that for any $\rho_a$-admissible curve $Y$ centered at 0, $F_*^{s_a^n(\omega)}(Y|_\omega)$ is still a $\rho_a$-admissible curve $Y$ centered at 0. From now on, we will only deal with admissible curves of this kind and simply call them **admissible curves** for short.

Finally, let us further introduce some useful functions compatible with $G_a$ and Markov partitions with respect to the natural time $n \in \mathbb{N}$. To begin with, given $x \in \mathbf{I}_{\beta_a}$, let

$$\iota_n(x) := \min\{k \in \mathbb{N} : s_a^k(x) \geq n\} = 1 + \max\{k \in \mathbb{N}_0 : s_a^k(x) < n\}.$$

Then let

$$\mathcal{P}_n := \bigcup_{k=1}^{\infty} \left\{ \omega \in Q_a^k : \iota_n(\omega) = k \right\} = \bigcup_{k=1}^{\infty} \left\{ \omega \in Q_a^k : s_a^{k-1}(\omega) < n \leq s_a^k(\omega) \right\},$$

Moreover, given $x \in \mathbf{I}_{\beta_a}$, let

$$\zeta_n(x) := s_a^{\iota_n(x)}(x) \quad \text{and} \quad \xi_n(x) := \begin{cases} s_a(G_a^{\iota_n(x)-1} x) & , \text{ if } \zeta_n(x) > n \\ 0 & , \text{ if } \zeta_n(x) = n \end{cases}.$$

Now let us state some simple and useful corollary of (4.1) and (4.2) for later usage.

**Lemma 4.1.1.** *There exists $\tilde{C}_a > 1$, determined by $Q_a$, such that the following statement holds. Let $i_1, i_2, \cdots, i_n$ be $\mathbb{N} \cup \{\infty\}$-valued functions on $\mathbf{I}_{\beta_a}$ compatible with $G_a$. If $i_{j+1} > i_j$ on* dom($i_j$) *for $j = 1, \cdots, n-1$, then we have:*

$$|\{x \in \mathbf{I}_{\beta_a} : \sum_{j=1}^{n} s_a(G_a^{i_j(x)}(x)) \geq Cn\}| \leq e^{-\frac{\gamma_a}{2}Cn}, \quad \forall n \in \mathbb{N}, C \geq \tilde{C}_a. \tag{4.3}$$

*Proof.* Denote the set to be estimated by $E$ and let $X_j := s_a \circ G_a^{i_j}$ for $1 \leq j \leq n$. Given $s_1, \cdots, s_n \in \mathbb{N}$, for every $1 \leq k \leq n$, there exists a subset $\Omega_k$ of $\cup_{i=0}^{\infty} Q_a^i$, such that the set

$$S_k := \{x \in \mathbf{I}_{\beta_a} : X_j \geq s_j, 1 \leq j \leq k\}$$

is the union of elements in $\Omega_k$ together with a measure zero set. From (4.1) and (4.2) we know that when $1 \leq k \leq n-1$,

$$\frac{|S_{k+1} \cap \omega|}{|\omega|} \leq 9 \cdot |\{x \in \mathbf{I}_{\beta_a} : s_a(x) \geq s_{k+1}\}| \leq 9\hat{C}_a e^{-\gamma_a \cdot s_{k+1}}, \quad \forall \omega \in \Omega_k.$$

It implies that

$$|\{x \in \mathbf{I}_{\beta_a} : X_j \geq s_j, 1 \leq j \leq n\}| = |S_n| \leq 9\hat{C}_a e^{-\gamma_a \cdot s_n} \cdot |S_{n-1}| \leq \cdots \leq e^{-\gamma_a(s_1 + \cdots + s_n) + C_0 n},$$

where $C_0 := \log(9\hat{C}_a) > 0$. Note that

$$E \subset \bigcup_{N \geq Cn} \bigcup_{\sum_{j=1}^{n} s_j = N} \{x \in \mathbf{I}_{\beta_a} : X_j = s_j, 1 \leq j \leq n\},$$

and

$$\#\{(s_1, \cdots, s_n) \in \mathbb{N}^n : \sum_{j=1}^{n} s_j = N\} = \binom{N-1}{n-1} \leq \binom{N}{n} \leq \left(\frac{eN}{n}\right)^n \leq (eC)^{C^{-1}N}.$$

It follows that

$$|E| \leq \sum_{N \geq Cn} \binom{N-1}{n-1} e^{-\gamma_a N + C_0 n} \leq \sum_{N \geq Cn} e^{(\gamma(C) - \gamma_a)N},$$

where

$$\gamma(C) := C^{-1}(\log C + C_0 + 1) \to 0, \quad \text{as} \quad C \to +\infty.$$

Then by choosing $\tilde{C}_a$ large, the conclusion follows. □

**Lemma 4.1.2.** *There exists $C'_a > 0$, such that for every $n \in \mathbb{N}$,*

$$|\{x \in \mathbf{I}_{\beta_a} : \xi_n(x) \geq s\}| \leq C'_a \cdot s e^{-\gamma_a s}. \tag{4.4}$$

*Proof.* Since $|\mathbf{I}_{\beta_a} \setminus \mathrm{dom}(G)| = 0$ and $G_a(\mathrm{dom}(G)) \subset \mathbb{I}_{\rho_a}$, by (4.1) and (4.2), it suffices to prove the lemma for $x \in \mathbb{I}_{\rho_a}$ instead of $x \in \mathbf{I}_{\beta_a}$. Consider the following tower system $(\Omega, H)$ induced by the Markov map $G_a$:

$$\Omega := \{(x, n) \in \mathbb{I}_{\rho_a} \times \mathbb{N}_0 : 0 \leq n < s_a(x)\} \subset \bigcup_{\omega \in Q_a} \omega \times \mathbb{N}_0,$$

$$H : \Omega \circlearrowleft, \quad (x, n) \mapsto \begin{cases} (x, n+1), & \text{if } n < s_a(x) - 1 \\ (G_a(x), 0), & \text{if } n = s_a(x) - 1 \end{cases}.$$

The Lebesgue measure Leb on $\mathbb{I}_{\rho_a}$ naturally induces a measure on $\Omega$, stilled denoted by Leb, as follows:

$$\mathrm{Leb}|_{\omega \times \{n\}} = \mathrm{Leb}|_{\omega}, \quad \forall \omega \in Q_a,\ 0 \leq n < s_a(\omega).$$

It is well known that $(\Omega, H)$ admits an invariant measure $\nu$ that is absolutely continuous with respect to Leb, and $\frac{\mathrm{d}\nu}{\mathrm{d}\,\mathrm{Leb}} \asymp 1$. See, for example, [You99].

Now given $n, s \in \mathbb{N}$, note that

$$x \in \mathbb{I}_{\rho_a} \text{ and } \xi_n(x) \geq s \Longrightarrow H^n(x, 0) \in \omega \times \mathbb{N} \text{ for some } \omega \in Q_a \text{ with } \omega \subset \mathbb{I}_{\rho_a} \text{ and } s_a(\omega) \geq s.$$

That is to say, for $E := \{x \in \mathbb{I}_{\rho_a} : \xi_n(x) \geq s\}$,

$$E \times \{0\} \subset H^{-n}\Big( \bigcup_{\substack{\omega \subset \mathbb{I}_{\rho_a}, \omega \in Q_a \\ s_a(\omega) \geq s}} \omega \times \{1, \cdots, s_a(\omega) - 1\} \Big).$$

Since $\nu$ is $H$-invariant and $\frac{\mathrm{d}\,\nu}{\mathrm{d}\,\mathrm{Leb}} \asymp 1$, the conclusion follows from the observation above and (4.1). $\square$

## 4.2 A technical proposition

Denote

$$M_\alpha := \left\lceil \frac{\log \frac{1}{\alpha}}{10 \log 2} \right\rceil.$$

Note that for $N_\alpha$ introduced in Lemma 2.3.1, we have $M_\alpha \leq \frac{1}{5} N_\alpha + 1$.

**Proposition 4.2.1.** *There exists $\beta_0 > 0$, such that when $\alpha > 0$ is sufficiently small, for any $M \in \mathbb{N}$ with $M_\alpha \leq M \leq 2M_\alpha$ and any admissible curve $Y : \mathbb{I}_{\rho_a} \to \mathbb{R}$, we have*

$$\textit{for} \quad E := \{x \in \mathbb{I}_{\rho_a} : |f_M(\hat{Y}(x))| \leq \alpha^{1-2\eta_b}\}, \quad |E| \leq \alpha^{\beta_0}. \tag{4.5}$$

The rest of this section is devoted to the proof of Proposition 4.2.1. Let us follow the argument in [BST03, GS14] with some modification. We may suppose that $E$ is nonempty, i.e. there exists $z_0 \in \hat{Y}(\mathbb{I}_{\rho_a})$, such that $|f_M(z_0)| \leq \alpha^{1-2\eta_b}$. For each $i \in \mathbb{N}$, denote $y_i := f_i(z_0)$ and $z_i = F^i(z_0)$. Recall the constant $\hat{A}_0 > 1$ appearing in Lemma 2.2.2 and the notation $B^+_\alpha(\cdot)$ introduced in § 2.2.2. Define

$$B_0 := \mathbf{I}(y_0, \hat{A}_0 \alpha) \quad \text{and} \quad B_{i+1} := B^+_\alpha(B_i), \quad \forall\, i \in \mathbb{N}_0.$$

Note that $Y(\mathbb{I}_{\rho_a}) \subset B_0$. Then by definition and (2.19),

$$|B_i| < 4^{i+1}\hat{A}_0\alpha, \quad Q_b(B_i) \subset B_{i+1} \quad \text{and} \quad f_i(\hat{Y}(\mathbb{I}_{\rho_a})) \subset B_i, \quad \forall i \in \mathbb{N}_0.$$

Also note that by our assumption, $|f_M(z_0)| \leq \alpha^{1-2\eta_b}$ and $M \leq 2M_\alpha < N_\alpha$, so from Lemma 2.3.1 we know that

$$|y_i| \geq 2\sqrt{\alpha} \quad \text{and} \quad \left|\partial_y f_{M-i}(z_i)\right| \geq C_b \sigma_b^{M-i}, \quad 0 \leq i < M. \tag{4.6}$$

As a result,

$$\sum_{i=0}^{M-1} \frac{|B_i|}{|y_i|} < 2\hat{A}_0\sqrt{\alpha} \sum_{i=0}^{2M_\alpha - 1} 4^i < \hat{A}_0\sqrt{\alpha} \cdot 4^{2M_\alpha} < 16\hat{A}_0\alpha^{\frac{1}{5}}.$$

Then according to (2.21), when $\alpha > 0$ is small,

$$\prod_{k=i}^{j} \sup_{y,y' \in B_k} \frac{|Q_b'(y)|}{|Q_b'(y')|} < 2, \quad 0 \leq i < j < M. \tag{4.7}$$

Following [BST03, GS14], let us define some notations and constants. Firstly, let

$$\lambda_j = |\partial_y f_{M-j}(z_j)| \cdot \sigma_b^{-\frac{M-j}{2}} \geq C_b \sigma_b^{\frac{M-j}{2}}, \quad 0 \leq j \leq M. \tag{4.8}$$

Note that by definition,

$$\lambda_i < 4^{j-i}\lambda_j, \quad 0 \leq i < j \leq M.$$

Secondly, recall $\hat{\delta}_2(\cdot)$ appearing in (3.13) and let

$$\epsilon_a := \hat{\delta}_2(\rho_a), \qquad \kappa := 128\epsilon_a^{-1}\left(\hat{A}_0 + (1 - \sigma_b^{-\frac{1}{2}})^{-1} + 1\right).$$

Then for every $x \in \mathbb{I}_{\rho_a}$, let

$$0 = t_1(x) < \tilde{t}_1(x) \leq \cdots \leq t_{q(x)}(x) < \tilde{t}_{q(x)}(x) \leq M \quad \text{and} \quad t_{q(x)+1}(x) \geq \tilde{t}_{q(x)}(x)$$

be all the times defined inductively in the following way:

$$\tilde{t}_i(x) := \max\{ s \in \mathbb{N} : t_i(x) < s \le M, \lambda_{t_i(x)} \le \kappa \lambda_s \}, \quad t_{i+1}(x) := \zeta_{\tilde{t}_i(x)}(x), \quad \forall 1 \le i \le q(x).$$

By definition, for every $x \in \mathbb{I}_{\rho_a}$, $1 \le q(x) < M$, and when $1 \le i \le q(x)$,

- for the unique element $\omega \in \mathcal{P}_{t_i(x)}$ containing $x$, $t_i \equiv t_i(x)$ and $\tilde{t}_i \equiv \tilde{t}_i(x)$ on $\omega$;
- $\frac{\kappa}{4}\lambda_{\tilde{t}_i} < \lambda_{t_i} \le \kappa\lambda_{\tilde{t}_i}$ and $\lambda_j < \lambda_{\tilde{t}_i}$ when $\tilde{t}_i < j \le M$.

Thirdly, given $x \in \mathbb{I}_{\rho_a}$, let

$$\hat{q}(x) := \max\left\{ 1 \le i < q(x) : \lambda_{\tilde{t}_i(x)} \ge \alpha^{-2\eta_b} \right\}.$$

**Lemma 4.2.2.** *The exist absolute constants $\theta > 0$ and $\beta > 0$, such that when $\alpha > 0$ is small, we have a measurable subset $E'$ of $E$ that satisfies*

$$|E \setminus E'| \le \alpha^{\beta} \quad \textit{and} \quad \hat{q}(x) \ge \theta \log\frac{1}{\alpha}, \ \forall x \in E'. \tag{4.9}$$

*Proof.* By definition, for every $x \in E$,

$$\alpha^{-2\eta_b} > \lambda_{\tilde{t}_{\hat{q}(x)+1}} = \lambda_0 \cdot \prod_{i=1}^{\hat{q}(x)+1} \frac{\lambda_{\tilde{t}_i(x)}}{\lambda_{t_i(x)}} \cdot \prod_{i=1}^{\hat{q}(x)} \frac{\lambda_{t_{i+1}(x)}}{\lambda_{\tilde{t}_i(x)}} \ge C_b \sigma_b^{\frac{M}{2}} \kappa^{-\hat{q}(x)-1} \cdot 4^{-\sum_{i=1}^{\hat{q}(x)}(t_{i+1}(x)-\tilde{t}_i(x))}.$$

Recall that

$$M \ge M_\alpha \ge \frac{\log\frac{1}{\alpha}}{10\log 2} \quad \text{and} \quad \eta_b = \frac{\log\sigma_b}{100\log 2},$$

so $C_b\kappa^{-1}\sigma_b^{\frac{M}{2}} \ge \alpha^{-4\eta_b}$, provided that $\alpha > 0$ is small. Therefore, for $\theta_1 := \frac{\eta_b}{\log\kappa}$ and $\theta_2 := \frac{\eta_b}{\log 4}$,

$$\hat{q}(x) \le \theta_1 \log\frac{1}{\alpha} \Longrightarrow \sum_{i=1}^{\hat{q}(x)} (t_{i+1}(x) - \tilde{t}_i(x)) \ge \theta_2 \log\frac{1}{\alpha}. \tag{4.10}$$

For every $\theta \in (0, \theta_1]$, denote

$$T_\theta := \bigcup_{q=1}^{\lfloor \theta \log \frac{1}{\alpha} \rfloor} \Big\{ x \in E : \hat{q}(x) = q, \sum_{i=1}^{q} (t_{i+1}(x) - \tilde{t}_i(x)) \geq \theta_2 \log \frac{1}{\alpha} \Big\}.$$

From Lemma 4.1.1 we know that when $C \geq \tilde{C}_a$,

$$\Big| \{ x \in E : \hat{q}(x) = q, \sum_{i=1}^{q} (t_{i+1}(x) - \tilde{t}_i(x)) \geq Cq \} \Big| \leq e^{-\frac{\gamma_a}{2} Cq}, \quad \forall\, q \in \mathbb{N}.$$

It follows that for $\theta := \theta_1 \wedge (\tilde{C}_a^{-1} \theta_2)$ and $\beta := \frac{\gamma_a \theta_2}{3}$,

$$|T_\theta| \leq \theta \log \frac{1}{\alpha} \times e^{-\frac{\gamma_a}{2} \cdot \theta_2 \log \frac{1}{\alpha}} \leq \alpha^\beta,$$

where the last inequality holds provided that $\alpha > 0$ is small. Fixing such a pair of $(\theta, \beta)$ and letting $E' := E \setminus T_\theta$, the conclusion follows from (4.10).

$\square$

Thanks to Lemma 4.2.2, we may prove the proposition for $E'$ instead of $E$. For each $1 \leq i \leq \lfloor \theta \log \frac{1}{\alpha} \rfloor$, define

$$\Omega_i := \{ \omega : \omega \in \mathcal{P}_{t_i(x)} \text{ for some } x \in E' \} \quad \text{and} \quad E_i := \bigcup_{\omega \in \Omega_i} \omega\,.$$

By definition, $E_1 \supset E_2 \supset \cdots \supset E_{\lfloor \theta \log \frac{1}{\alpha} \rfloor} \supset E'$, and the proof of Proposition 4.2.1 will be done once we have proved:

**Lemma 4.2.3.** *For each $1 \leq i < \lfloor \theta \log \frac{1}{\alpha} \rfloor$ and each $\omega_i \in \Omega_i$,*

$$\sum_{\substack{\omega \subset \omega_i \\ \omega \in \Omega_{i+1}}} |\omega| \leq \frac{27}{28} |\omega_i|.$$

*Proof.* Fixing $\omega_i \in \Omega_i$, let $t_i = t_i|_{\omega_i}$ and $\tilde{t}_i = \tilde{t}_i|_{\omega_i}$, both of which are constants. Denote

$$\Gamma := \{Q_a^{t_i}(\omega) : \omega \in \Omega_{i+1},\ \omega \subset \omega_i\} \subset \mathcal{Q}_a.$$

In view of (4.2), the conclusion of the lemma is reduced to

$$\sum_{\omega \in \Gamma} |\omega| \leq \frac{3}{2}\rho_a. \tag{4.11}$$

Note that $Y_i := F_*^{t_i}(Y|_{\omega_i})$ is an admissible curve. For every $\omega \in \mathcal{Q}_a$ containing in $\mathbb{I}_{\rho_a}$, denote $Z_\omega^{\pm} = F_*(Y_i|_{\pm\omega})$. Moreover, let

$$\Gamma' := \{\omega \in \Gamma : \|Z_\omega^+ - Z_\omega^-\|_{Q_a(\omega)} \geq \epsilon_a \alpha\}.$$

From (3.13) for $r = \rho_a$ we know that $\sum_{\omega \in \Gamma \setminus \Gamma'} |\omega| \leq \rho_a$, so (4.11) is further reduced to

$$\omega \in \Gamma' \Longrightarrow -\omega \notin \Gamma. \tag{4.12}$$

Now given $\omega \in \Gamma'$, for $\tilde{t}_i \leq j \leq M$, let us define

$$\Delta_j := \inf_{x^\pm \in \pm\omega} |f_{j-t_i}(\hat{Y}_i(x^+)) - f_{j-t_i}(\hat{Y}_i(x^-))|.$$

By definition, to show $-\omega \notin \Gamma$, we only need to show that $\Delta_M > 2\alpha^{1-2\eta_b}$. To this end, let us estimate the lower bound of $\Delta_{\tilde{t}_i}$ first. Denote $n := \tilde{t}_i - t_i - 1$ temporarily. Since $\omega \in \Gamma'$, by intermediate value theorem and (4.7),

$$\sup_{x \in \omega} |f_{n+1}(\hat{Y}_i(x)) - f_{n+1}(\hat{Y}_i(-x))| = \sup_{x \in Q_a(\omega)} |f_n(\hat{Z}_\omega^+(x)) - f_n(\hat{Z}_\omega^-(x))| \geq \frac{\epsilon_a \alpha}{2} |\partial_y f_n(z_{t_i+1})|,$$

where

$$|\partial_y f_n(z_{t_i+1})| > 4^{-1}|\partial_y f_{n+1}(z_{t_i})| = \frac{\lambda_{t_i}}{4\lambda_{\tilde{t}_i}} \cdot \sigma_b^{\frac{\tilde{t}_i - t_i}{2}} > \frac{\lambda_{t_i}}{4\lambda_{\tilde{t}_i}} > \frac{\kappa}{16}.$$

Recall that $Y_i$ is an admissible curve and $s_a(\omega) \geq n+1$. Then according to Lemma 2.2.2, we have:

$$\max_{S=\pm\omega} \sup_{x,x' \in S} |f_{n+1}(\hat{Y}_i(x)) - f_{n+1}(\hat{Y}_i(x'))| \leq 2\hat{A}_0 \alpha.$$

Combining the three preceding displayed inequalities, we have:

$$\Delta_{\tilde{t}_i} > (\frac{\kappa\epsilon_a}{32} - 4\hat{A}_0)\alpha. \tag{4.13}$$

Recall (2.18) and compare it with definition of $\Delta_j$. It follows that for $D_j := \min_{y \in B_j} |Q_b'(y)|$, $\tilde{t}_i \leq j < M$, we have:

$$\Delta_M \geq \Delta_{\tilde{t}_i} \prod_{j=\tilde{t}_i}^{M-1} D_j - 2\alpha \left(1 + \sum_{j=\tilde{t}_i+1}^{M-1} \left(\prod_{l=j}^{M-1} D_l\right)\right).$$

Due to (4.7) and (4.8), and noting that $\lambda_j < \lambda_{\tilde{t}_i}$ when $j > \tilde{t}_i$, we have:

$$\frac{1}{2}\lambda_j \sigma_b^{\frac{M-j}{2}} \leq \prod_{l=j}^{M-1} D_l \leq \lambda_j \sigma_b^{\frac{M-j}{2}} < \lambda_{\tilde{t}_i} \sigma_b^{\frac{M-j}{2}}.$$

Combing the two displayed inequalities above, we obtain that

$$\Delta_M > \lambda_{\tilde{t}_i} \sigma_b^{\frac{M-\tilde{t}_i}{2}} (\Delta_{\tilde{t}_i}/2 - 2\alpha(1 - \sigma_b^{-\frac{1}{2}})^{-1}).$$

Substituting (4.13) into the inequality above and making use of $\lambda_{\tilde{t}_i} \geq \alpha^{-2\eta_b}$ and the choice of $\kappa$, we have:

$$\Delta_M > 2\alpha^{1-2\eta_b}.$$

That is to say, $\pm\omega$ cannot both intersect $\Gamma$, which completes the proof of the lemma. □

## 4.3 The vertical slow recurrence condition

**Proposition 4.3.1.** *When $\alpha > 0$ is sufficiently small, for every $\varepsilon \in (0, 1)$, there exists $\delta = \delta(\varepsilon) > 0$, depending only on $\varepsilon$, such that following inequality holds for $n \in \mathbb{N}$ large.*

$$\mathrm{Leb}\left(\left\{(x, y) \in \mathbf{I}_{\beta_a} \times \Lambda_b : \sum_{\substack{0 \le i < n \\ |f_i(x,y)| < \delta\alpha^{1-2\eta_b}}} \log \frac{1}{|f_i(x, y)|} > \varepsilon n\right\}\right) \le e^{-n^{\frac{1}{2} - \frac{1}{\sqrt{\log\log n}}}}, \tag{4.14}$$

*As a consequence, for Lebesgue a.e. $(x, y) \in \mathbf{I}_{\beta_a} \times \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{0 \le i < n \\ |f_i(x,y)| < \delta\alpha^{1-2\eta_b}}} \log \frac{1}{|f_i(x, y)|} \le \varepsilon. \tag{4.15}$$

This section is devoted to the proof of Proposition 4.3.1. The argument is based on Proposition 3.4.4 and Proposition 4.2.1. Let us summarize and reformulate the results in those two propositions in the following lemma first.

**Lemma 4.3.2.** *The following statement holds when $\alpha > 0$ is small. Let $Y : \mathrm{dom}(Y) \to \Lambda_b$ be either an admissible curve with $\mathrm{dom}(Y) = \mathbb{I}_{\rho_a}$ or a constant curve with $\mathrm{dom}(Y) = \mathbf{I}_{\beta_a}$, and suppose $q, n \in \mathbb{N}$ satisfy that either $e^{-q} \le \alpha^{1-2\eta_b}$ and $n \ge 2M_\alpha$, or $e^{-q} < \alpha^2$ and $n$ is arbitrary. Then we have:*

$$|\{x \in \mathrm{dom}(Y) : |f_n(\hat{Y}(x))| \le e^{-q}\}| < e^{-q^{1-\frac{1}{(\log\log q)^{0.6}}}}. \tag{4.16}$$

*Proof.* Denote the set to be estimated in (4.16) by $E$. Let us consider two complementary situations according to $e^{-q} \le \alpha^2$ or not.

If $\alpha^2 \le e^{-q} \le \alpha^{1-2\eta_b}$ and $n \ge 2M_\alpha$, then $q \asymp M_\alpha \asymp \log\frac{1}{\alpha}$. Let $\tilde{n} := n - 2M_\alpha$. By

Lemma 4.1.2, when $\alpha > 0$ is small,

$$|\{x \in \mathrm{dom}(Y) : \xi_{\tilde{n}}(x) > M_\alpha\}| < C'_a M_\alpha e^{-\gamma_a M_\alpha} < e^{-q^{1-\frac{1}{\sqrt{\log q}}}}.$$

Therefore, it suffices to consider $E' := \{x \in E : \xi_{\tilde{n}}(x) \leq M_\alpha\}$ instead of $E$. For $\omega \in \mathcal{P}_{\tilde{n}}$, if $\omega \cap E' \neq \emptyset$, then $\xi_{\tilde{n}}(\omega) \leq M_\alpha$ and hence for $n' := n - \zeta_{\tilde{n}}(\omega)$, $M_\alpha \leq n' \leq 2M_\alpha$. Therefore, we can apply Proposition 4.2.1 to the admissible curve $Z := F_*^{\zeta_{\tilde{n}}(\omega)}(Y|_\omega)$ with $M = n'$, which implies that, provided that $\alpha > 0$ is small,

$$|\{x \in \mathbb{I}_{\rho_a} : |f_{n'}(\hat{Z}(x))| \leq e^{-q}\}| \leq \alpha^{\beta_0} \leq e^{-\frac{\beta_0}{2}q}.$$

The desired estimate of $|E'|$ follows from the inequality above and (4.2).

If $e^{-q} < \alpha^2$, the argument is similar and the main difference is to apply Proposition 3.4.4 instead of Proposition 4.2.1. Firstly, by Lemma 4.1.2 again, we only need to consider $E' := \{x \in E : \xi_n(x) \leq q^{1-\frac{1}{(\log\log q)^{0.65}}}\}$ instead of $E$. Secondly, by (4.2), it suffices to prove the following statement. Let $Z : \mathrm{dom}(Z) \to \Lambda_b$ be either an admissible curve with $\mathrm{dom}(Z) = \mathbb{I}_{\rho_a}$ or a constant curve with $\mathrm{dom}(Z) = \mathbf{I}_{\beta_a}$. If $\omega \in \mathcal{Q}_a$ and $n \in \mathbb{N}$ satisfy that $\omega \subset \mathrm{dom}(Z)$ and $n < s_a(\omega) \leq q^{1-\frac{1}{(\log\log q)^{0.65}}}$, then

$$|\{x \in \omega : |f_n(\hat{Z}(x))| \leq e^{-q}\}| \leq |\omega| \cdot e^{-q^{1-\frac{1}{(\log\log q)^{0.65}}}}. \tag{4.17}$$

On the one hand, Proposition 3.4.4 is applicable here, because $|Q_a^n(\omega)| < \hat{r}_a$ holds by the choice of our Markov map $G_a$. Then the left hand side in (4.17) is bounded by

$$|\{x \in \omega : |f_n(\hat{Z}(x))| \leq e^{-\frac{q}{2}}\alpha\}| \leq \begin{cases} e^{-\frac{q}{2\hat{n}_a(r)}} & n \leq \hat{n}_a(r) \\ e^{-\frac{q}{2}\cdot n^{-\frac{1}{(\log\log n)^{0.7}}}} & n > \hat{n}_a(r) \end{cases}.$$

Since $n < q$, in both cases it is bounded by $e^{-\frac{1}{2}\cdot q^{1-\frac{1}{(\log\log q)^{0.7}}}}$, provided that $\alpha$ is small and hence $q$ is large accordingly. On the other hand, because $|G_a(\omega)| = 2\rho_a$ and $\|G_a'\|_\omega \leq 4^{s_a(\omega)}$, $s_a(\omega) \leq q^{1-\frac{1}{(\log\log q)^{0.65}}}$ implies that $|\omega| > e^{-q^{1-\frac{1}{(\log\log q)^{0.7}}}}$ when $q$ is large. (4.17) follows. □

Now we are ready to prove Proposition 4.3.1. Let us start with some definitions of notations. Given $\delta > 0$, let $\hat{q}(\delta) := \left\lceil \log \frac{1}{\delta\alpha^{1-2\eta_b}} \right\rceil$. Fixing an arbitrary $y_0 \in \Lambda_b$, let

$$q_k(x) := \left\lfloor \log \frac{1}{|f_k(x, y_0)|} \right\rfloor \quad \text{and} \quad q_k^\delta(x) = \begin{cases} q_k(x) & , \text{ if } \; q_k(x) \geq \hat{q}(\delta) \\ 0 & , \text{ if } \; q_k(x) < \hat{q}(\delta) \end{cases}.$$

Moreover, for every $K \in \mathbb{N}$, define

$$E_K = E_K(\varepsilon, \delta, y_0) := \{x \in \mathbf{I}_{\beta_a} : \sum_{k=1}^{K} q_k^\delta(x) \geq \varepsilon K\}.$$

Then by Fubini's theorem, Proposition 4.3.1 is reduced to

**Lemma 4.3.3.** *The following statement holds when $\alpha > 0$ is small. For every $\varepsilon > 0$, there exists $\delta > 0$, depending only on $\varepsilon$, and there exists $K_0 = K_0(\alpha, \varepsilon) \in \mathbb{N}$, independent of $y_0 \in \Lambda_b$, such that*

$$|E_K| \leq e^{-K^{\frac{1}{2}-\frac{5}{(\log\log K)^{0.6}}}}, \quad \forall K \geq K_0. \tag{4.18}$$

The rest of this section is for the proof of Lemma 4.3.3. Given $K \in \mathbb{N}$ large, denote $\rho(K) := \left\lfloor \frac{\sqrt{K}}{(\log K)^2} \right\rfloor$. Decompose $E_K$ into a union $E_K^1 \cup E_K^2$, where

$$E_K^1 := \{x \in E_K : q_k(x) \vee \xi_k(x) \leq \rho(K),\ 1 \leq k \leq K\}, \quad E_K^2 := E_K \setminus E_K^1.$$

It follows immediately from Lemma 4.1.2 and Lemma 4.3.2 that when $K$ is large,

$$|E_K^2| < K \cdot \left(9\hat{C}_a e^{-\gamma_a \cdot \rho(K)} + e^{-\rho(K)^{1-\frac{1}{(\log\log\rho(K))^{0.6}}}}\right) < e^{-K^{\frac{1}{2}-\frac{2}{(\log\log K)^{0.6}}}},$$

so let us focus on the estimate of $|E_K^1|$.

To begin with, let $L := \lfloor \sqrt{K} \rfloor$ and for $l = 0, 1, \cdots, L-1$, let

$$\mathcal{M}_l := \{2M_\alpha \le k \le K : k \equiv l \mod L\} \quad \text{and} \quad E^1_{K,l} := \Big\{x \in E^1_K : \sum_{k \in \mathcal{M}_l} q^\delta_k(x) \ge \frac{\varepsilon\sqrt{K}}{2}\Big\}.$$

By definition, when $K$ is large, say $K \ge \left(\frac{M_\alpha}{\varepsilon}\right)^2$, $\#\mathcal{M}_l \ge \sqrt{K} - 1$ for $0 \le l \le L-1$ and

$$x \in E^1_K \Longrightarrow \sum_{l=0}^{L-1} \sum_{k \in \mathcal{M}_l} q^\delta_k(x) \ge \sum_{k=1}^{K} q^\delta_k(x) - 2M_\alpha \cdot \rho(K) \ge \frac{\varepsilon K}{2} \Longrightarrow x \in \bigcup_{l=0}^{L-1} E^1_{K,l},$$

so let us fix an arbitrary $0 \le l \le L-1$ and estimate the size of $E^1_{K,l}$. To begin with, we have the following lemma.

**Lemma 4.3.4.** *For every sequence of natural numbers $k_1 < \cdots < k_n$ in $\mathcal{M}_l$ and every $\hat{q}(\delta) \le q \le \rho(K)$, we have*

$$|\{x \in E^1_{K,l} : q^\delta_{k_j}(x) \ge q, j = 1, \cdots, n\}| \le e^{-n\, q^{1 - \frac{2}{(\log\log q)^{0.6}}}}. \tag{4.19}$$

*Proof.* Denote the set to be estimated in by $S$. Given $x \in S$ and $1 \le j \le n$, let $\omega_j(x)$ be the unique element in $\mathcal{P}_{k_j}$ containing $x$. Let $\Omega_0 := \{\mathbf{I}_{\beta_a}\}$ and let

$$\Omega_j := \{\omega_j(x) : x \in S\} \quad \text{and} \quad S_j := \bigcup_{\omega \in \Omega_j} \omega, \quad 1 \le j \le n.$$

By definition, $S_1 \supset S_2 \supset \cdots \supset S_n \supset S$, so it suffices to show that when $\hat{q}(\delta) \le q \le \rho(K)$, for each $0 \le j < n$ and each $\omega_j \in \Omega_j$,

$$|S_{j+1} \cap \omega_j| \le |\omega_j| \cdot e^{-q^{1 - \frac{2}{(\log\log q)^{0.6}}}}. \tag{4.20}$$

Fix $\omega_j \in \Omega_j$. When $j = 0$, denote $s = 0$ and $t = k_1 > 2M_\alpha$. When $j \ge 1$, denote

$s := \zeta_{k_j}(\omega_j)$ and $t := k_{j+1} - s$, and we have:

$$t = (k_{j+1} - k_j) - (s - k_j) \geq L - \xi_{k_j}(\omega_j) \geq \sqrt{K} - 1 - \rho(K) > \log K \cdot q > 2M_\alpha.$$

Note that for $Y_0 \equiv y_0$, $Z := F_*^s(Y_0|_{\omega_j})$ is either $Y$ itself when $j = 0$, or an admissible curve when $j \geq 1$. Given $\omega \in \Omega_{j+1}$ with $\omega \subset \omega_j$, by definition, there exists $x_\omega \in S_{j+1}$, such that $\omega_{j+1}(x_\omega) = \omega$. From $q_{k_{j+1}}(x_\omega) \geq q$ we know that for $y_\omega := f_{k_{j+1}}(x_\omega, y_0)$, $|y_\omega| \leq e^{-q}$. Then from Lemma 2.2.2 we know that

$$\sup_{x \in Q_a^s(\omega)} |f_t(\hat{Z}(x)) - y_\omega| \leq \hat{A}_0 \lambda_a^{-t} \alpha < e^{-q} \Longrightarrow \sup_{x \in Q_a^s(\omega)} |f_t(\hat{Z}(x))| < e^{-q+1}.$$

Since the inequality hold for every $\omega \in \Omega_{j+1}$ with $\omega \cap \Omega_{j+1} \neq \emptyset$, we have:

$$Q_a^s(\omega_j \cap S_{j+1}) \subset \{x \in \mathrm{dom}(Z) : |f_t(\hat{Z}(x))| \leq e^{-q+1}\}.$$

Then by Lemma 4.3.2 and (4.2), (4.20) follows and the proof is completed.

□

Now let us return to the proof of Lemma 4.3.3. From now on we will fix $\delta := e^{-\frac{2e^2}{\epsilon}-2}$ in the definition of $\hat{q}(\delta)$. Then let $t_0 := \lfloor \log \hat{q}(\delta) \rfloor$ and for every $K \in \mathbb{N}$, let $t_K := \lceil \log \rho(K) \rceil$. When $K$ is large and $t \leq t_K$, $t^2 e^t < \sqrt{K}$, so

$$c(t) := \left\lceil \sqrt{K} t^{-2} e^{-t} \right\rceil < \sqrt{K} t^{-2} e^{-t+1}, \quad \forall t_0 \leq t \leq t_K.$$

Moreover, let

$$E_{K,l}^1(t) = \bigcup_{\substack{k_1, \cdots, k_{c(t)} \in \mathcal{M}_l \\ k_1 < \cdots < k_{c(t)}}} \{x \in E_{K,l}^1 : e^t \leq q_{k_j}^\delta(x) < e^{t+1}, j = 1, \cdots, c(t)\}.$$

Given $x \in E_{K,l}^1$, note that for every $1 \leq k \leq K$, by the choice of $t_0$, either $q_k^\delta(x) = 0$ or

$q_k^\delta(x) \geq e^{t_0}$; by the choice of $t_K$, $q_k^\delta(x) \leq e^{t_K}$. Also note that $t_0 \geq \frac{2e^2}{\epsilon} + 1$ . Then the inequality

$$\sum_{t=t_0}^{t_K} e^{t+1} c(t) \leq \sum_{t=t_0}^{\infty} e^{t+1} \sqrt{K} t^{-2} e^{-t+1} < \frac{e^2 \sqrt{K}}{t_0 - 1} \leq \frac{\varepsilon \sqrt{K}}{2} \leq \sum_{k \in \mathcal{M}_l} q_k^\delta(x)$$

implies that there exists $t_0 \leq t \leq t_K$, such that

$$\#\{k \in \mathcal{M}_l : e^t \leq q_k^\delta(x) < e^{t+1}\} \geq c(t).$$

That is to say,

$$E^1_{K,l} = \bigcup_{t=t_0}^{t_K} E^1_{K,l}(t).$$

Due to (4.19), given $1 \leq k_1 < \cdots < k_{c(t)} \leq K$ in $\mathcal{M}_l$,

$$|\{x \in E^1_K(t) : q^\delta_{k_j}(x) \geq e^t, j = 1, \cdots, c(t)\}| \leq e^{-c(t)\, e^{t - \frac{2t}{(\log t)^{0.6}}}}, \quad \forall t \geq t_0.$$

Also note that for any pair of natural numbers $m \leq n$, we have

$$\binom{n}{m} \leq \frac{n^m}{m!} < \left(\frac{en}{m}\right)^m.$$

Therefore, for $n = \#\mathcal{M}_l < e\sqrt{K}$ and $m = c(t)$, when $K$ is large and $t_0 \leq t \leq t_K$,

$$|E^1_{K,l}(t)| \leq \binom{\#\mathcal{M}_l}{c(t)} \left(e^{-e^{t - \frac{2t}{(\log t)^{0.6}}}}\right)^{c(t)} < \left(\frac{\sqrt{K} e^{2 - e^{t - \frac{2t}{(\log t)^{0.6}}}}}{c(t)}\right)^{c(t)} < e^{-c(t) \cdot e^{t - \frac{3t}{(\log t)^{0.6}}}} < e^{-\sqrt{K} e^{-\frac{4t}{(\log t)^{0.6}}}}.$$

Since $t < t_K < \log K$, $e^{-\frac{4t}{(\log t)^{0.6}}} > K^{-\frac{4}{(\log \log K)^{0.6}}}$, so $|E^1_{K,l}(t)| < e^{K^{\frac{1}{2} - \frac{4}{(\log \log K)^{0.6}}}}$. Then

$$|E_K^1| \leq \sum_{l=0}^{L-1} \sum_{t=t_0}^{t_K} |E_{K,l}^1(t)| < \sqrt{K} \cdot \log K \cdot e^{K^{\frac{1}{2} - \frac{4}{(\log\log K)^{0.6}}}},$$

which completes the proof of Lemma 4.3.3 and consequently the proof of Proposition 4.3.1.

Chapter 5

# Proof of the main theorem

## 5.1 Positive vertical Lyapunov exponent

As shown in [Via97, GS14], that the vertical lower Lyapunov exponent is positive almost everywhere can be deduced from Proposition 4.3.1 and Lemma 2.3.1. More precisely, we have:

**Proposition 5.1.1.** *When $\alpha > 0$ is sufficiently small, for Lebesgue almost every $(x, y) \in \mathbf{I}_{\beta_a} \times \mathbb{R}$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log |\partial_y f_n(x, y)| \geq \frac{\eta_b}{2} \log \sigma_b > 0. \tag{5.1}$$

*Proof.* According to Proposition 4.3.1, we only need to consider an arbitrary point $(x, y)$ for which (4.15) holds. Denote the $F$-orbit of $(x, y)$ by $\{(x_i, y_i)\}_{i\in\mathbb{N}_0}$. Given $n \in \mathbb{N}$, let $0 \leq \nu_1 < \cdots < \nu_s \leq n$ be all the times $\nu$ such that $|f_\nu(x, y)| \leq \sqrt{\alpha}$. According to Lemma 2.3.1, we have:

- $\nu_{i+1} - \nu_i \geq N_\alpha$ for $1 \leq i < s$, and in particular $n \geq (s - 1)N_\alpha$ ;
- $|\partial_y f_{N_\alpha}(x_{\nu_i}, y_{\nu_i})| \geq |y_{\nu_i}|\alpha^{-1+\eta_b}$ for $1 \leq i < s$ ;
- $|\partial_y f_{\nu_{i+1}-\nu_i-N_\alpha}(x_{\nu_i+N_\alpha}, y_{\nu_i+N_\alpha})| \geq C_b\sigma_b^{\nu_{i+1}-\nu_i-N_\alpha}$ for $1 \leq i < s$ ;

- $|\partial_y f_{\nu_1}(x_0, y_0)| \geq C_b \sigma_b^{\nu_1}$ and $|\partial_y f_{n-\nu_s}(x_{\nu_s}, y_{\nu_s})| \geq C_b |y_{\nu_s}| \sqrt{\alpha} \sigma_b^{n-\nu_s}$.

Given $\varepsilon > 0$, let $\delta = \delta(\varepsilon)$ be determined in Proposition 4.3.1. Then by (4.15), when $n$ is sufficiently large, we have:

$$\prod_{i=1}^{s} |y_{\nu_i}| \geq \delta^s \alpha^{(1-2\eta_b)s} e^{-2\varepsilon n}.$$

Therefore,

$$|\partial_y f_n(x, y)| \geq C_b^{s+1} \delta^s \alpha^{\frac{3}{2}-\eta_b(s+1)} \sigma_b^{n-(s-1)N_\alpha} e^{-2\varepsilon n}.$$

Recall that due to the choice of $\sigma_b$ and $N_\alpha$ in Lemma 2.3.1,

$$\sigma_b^{N_\alpha} \leq \alpha^{-1} \Longrightarrow \alpha^{-\eta_b s} \sigma_b^{n-(s-1)N_\alpha} \geq \sigma_b^{\eta_b n}.$$

Also recall that $\delta$ depends only on $\varepsilon$ and $(s-1)N_\alpha \leq n$. Then for $\varepsilon = \frac{\eta_b}{6} \log \sigma_b$, summarizing the known facts above, we can conclude that when $\alpha > 0$ is small enough, $|\partial_y f_n(x, y)| \geq \sigma_b^{\frac{\eta_b}{2} n}$ for $n$ large. The proof is completed.

□

Now we have shown that the vertical lower Lyapunov exponent of $F$ has a positive lower bound. Also note that $Q_a$ has a unique ergodic a.c.i.p. $\mu_a$ on $\mathbf{I}_{\beta_a}$. Then, as a consequence of [AS11, Corollary 1.1], we can conclude that $F$ admits finitely many ergodic a.c.i.p.'s, and the union of their basins has full measure in $\mathbf{I}_{\beta_a} \times \Lambda_b$.

## 5.2 Number of ergodic a.c.i.p.'s

In this section, let us prove the last part of the main theorem. We adopt the approach in [ABV00, § 5] to prove the proposition below. Recall the notations $\hat{p}_c$, $\check{I}_c$ and $\check{I}_c^k = Q_c^k(\check{I}_c)$ introduced in § 2.1.2.

**Proposition 5.2.1.** *The following statement holds when $\alpha > 0$ is small. For every a.c.i.p. $\mu$ of $F$, there exists a rectangle $R \subset \mathbb{R}^2$, such that* $\mathrm{Leb}(R \setminus \mathcal{B}(\mu)) = 0$. *Here $\mathcal{B}(\mu)$ denotes the basin of $\mu$, and $R = \check{I}_a \times J$ satisfies that $|\check{I}_b^k \setminus J| \leq \alpha^{0.9}$ for some $k \in \mathbb{N}_0$.*

*Remark.* Following the approach in [ABV00, § 5], with little extra work than what we present below, one may provide an alternative proof for the existence of finitely many ergodic a.c.i.p.'s which does not rely on the result in [AS11].

Once the proposition is proved, the last part of the main theorem follows easily as a corollary below.

**Corollary 5.2.2.** *When $\alpha > 0$ is small, the number of ergodic a.c.i.p.'s of $F$ does not exceed that of $Q_a \times Q_b$, which equals $\hat{d} :=$* $\gcd(\hat{p}_a, \hat{p}_b)$. *Moreover, when $b \notin \mathscr{C}_{MT}$, the number of ergodic a.c.i.p.'s of $F$ is precisely $\hat{d}$.*

*Proof.* Let us follow the notations in Lemma 2.1.3 and further denote

$$S_k := \bigcup_{n=0}^{\hat{m}-1} \check{I}_a^n \times \check{I}_b^{n+k}, \quad 0 \leq k < \hat{d} \Longrightarrow \bigcup_{k=0}^{\hat{d}-1} S_k = \mathrm{supp}(\mu_a) \times \mathrm{supp}(\mu_b),$$

where $\hat{m} := \mathrm{lcm}(\hat{p}_a, \hat{p}_b)$. On the one hand, from Lemma 2.1.3 we know that for $F_0 := Q_a \times Q_b$, $F_0 : S_k \circlearrowleft$ is topologically transitive and admits a unique ergodic a.c.i.p. of full support for every $0 \leq k < \hat{d}$. Therefore, the number of ergodic a.c.i.p.'s of $F_0$ coincides with $\hat{d}$. On the other hand, according to Proposition 5.2.1, provided that $\alpha > 0$ is small enough, if $\mu$ is an a.c.i.p. of $F$, there exists a rectangle $R = \check{I}_a \times J$ with $|\check{I}_b^k \setminus J| \leq \alpha^{0.9}$ for some $k \in \mathbb{N}_0$, such that $\mathrm{Leb}(R \setminus \mathcal{B}(\mu)) = 0$. Then it follows that $\mathrm{Leb}(S_k \setminus \mathcal{B}(\mu)) \leq C\alpha^{0.9}$ for some constant $C > 1$ depending only on $(a, b)$, which implies that, in particular, for every ergodic a.c.i.p. $\mu$ of $F$, there exists $0 \leq k < \hat{d}$, such that $\mathrm{Leb}(S_k \cap \mathcal{B}(\mu)) > \frac{1}{2}\mathrm{Leb}(S_k)$. That is to say, $F$ admits at most $\hat{d}$ ergodic a.c.i.p.'s.

When $b \notin \mathscr{C}_{MT}$, we know that for the minimal restrictive interval $\hat{I}_b$ of $Q_b$, we have $Q_b^{\hat{p}_b}(\hat{I}_b) \subset \mathrm{int}\, \hat{I}_b$. It means that there exist a sequence of closed intervals $\{J_b^k : k \in \mathbb{N}_0\}$ with

$J_b^0 = \hat{I}_b$, such that $Q_b(J_b^{k+\hat{p}_b}) = J_b^k$ and $Q_b(J_b^k) \subset \operatorname{int} J_b^{k+1}$ for $k \in \mathbb{N}_0$. Thus by continuity, when $\alpha > 0$ is small, for

$$\hat{S}_k := \bigcup_{n=0}^{\hat{m}-1} \check{I}_a^n \times J_b^{n+k} \supset S_k, \quad 0 \le k < \hat{d},$$

they are all $F$-invariant. Since restricted to each of them, $F$ admits an a.c.i.p., and since the intersection of each two of them has zero Lebesgue measure, in this case $F$ has exactly $\hat{d}$ ergodic a.c.i.p.'s. □

Now we turn to the proof of Proposition 5.2.1. By mimicking the argument in [ABV00, § 5], we define "hyperbolic time" in our situation as below.

**Definition 5.2.1.** Fix $r > 0$ and $\sigma > 1$. Given $(x, y) \in \mathbf{I}_{\beta_a} \times \Lambda_b$, $n \in \mathbb{N}$ is called an $(r, \sigma)$**-hyperbolic time** of $(x, y)$, if $(x, n)$ is an $r$-admissible pair and the following holds:

$$|f_{n-k}(x, y)| \ge \sigma^{-\frac{k}{2}} \wedge \alpha^{1-\frac{5}{3}\eta_b} \quad \text{and} \quad |\partial_y f_k(F^{n-k}(x, y))| \ge \sigma^k, \quad 1 \le k \le n. \tag{5.2}$$

Given the definition of hyperbolic time, Lemma 5.2.3 below is parallel to [ABV00, Lemma 5.2, Corollary 5.3] and Lemma 5.2.5 below is parallel to [ABV00, Lemma 5.4].

**Lemma 5.2.3.** *Given $\sigma > 1$, the following holds when $\alpha > 0$ is small. Suppose $n \in \mathbb{N}$ is an $(r, \sigma)$-hyperbolic time of $(x_0, y_0)$ for some $r \in (0, \hat{r}_a)$. Denote $(x_k, y_k) = F^k(x_0, y_0)$ for $1 \le k \le n$. Then $F^n$ maps an open neighborhood of $(x_0, y_0)$ onto $\mathbb{I}(x_n, r) \times \mathbb{I}(y_n, \alpha^{1-\frac{4}{3}\eta_b})$ diffeomorphically with distortion bounded by* 10.

*Proof.* Denote $I_k := \operatorname{comp}(x_k, n-k, r)$, $J_k := \mathbb{I}(y_k, 3\alpha^{1-\frac{4}{3}\eta_b}\sigma^{k-n})$ and $R_k := I_k \times J_k$ for $0 \le k \le n$. Let $G_n := \mathbb{I}(x_n, r) \times \mathbb{I}(y_n, \alpha^{1-\frac{4}{3}\eta_b})$ and let $G_k$ be the connected component of $F^{-(n-k)}(G_n)$ containing $(x_k, y_k)$ for $0 \le k < n$. We claim that $G_k \subset R_k$ for $0 \le k \le n$. When $k = n$, there is nothing to prove. By induction, suppose that for some $0 < m \le n$, $G_k \subset R_k$

has been proved for $m \le k \le n$. Since $F^k(G_m) \subset G_{m+k} \subset R_{m+k}$ for $0 \le k \le n-m$, and since

$$\theta := \sum_{k=m}^{n-1} \frac{|J_k|}{|y_k|} \le \sum_{k=m}^{n-1} \frac{6\alpha^{1-\frac{4}{3}\eta_b}\sigma^{k-n}}{\alpha^{1-\frac{5}{3}\eta_b} \wedge \sigma^{\frac{k-n}{2}}} < 6\sum_{k=1}^{\infty}\left(\alpha^{\frac{\eta_b}{3}}\sigma^{-k} + \alpha^{1-\frac{4}{3}\eta_b}\sigma^{-\frac{k}{2}}\right) < \frac{7\alpha^{\frac{\eta_b}{3}}}{\sigma^{\frac{1}{2}}-1},$$

from (2.21) we know that

$$\sup_{p,q\in G_m\cap R_m} \frac{|\partial_y f_{n-m}(p)|}{|\partial_y f_{n-m}(q)|} \le e^{\frac{2\theta}{1-\theta}} < \frac{10}{9}, \tag{5.3}$$

where the last inequality holds provided that $\alpha > 0$ is small. Recall that by Lemma 2.2.2,

$$|f_{n-m}(x, y_m) - f_{n-m}(x_m, y_m)| \le \hat{A}_0\alpha < \alpha^{1-\frac{4}{3}\eta_b}, \quad \forall x \in I_m;$$

in particular, $f_{n-m}(I_m \times \{y_m\}) \subset G_n$. It follows that for every $(x, y) \in G_m \cap R_m$,

$$|y - y_m| \le \frac{10}{9} \cdot \frac{|f_{n-m}(x, y) - f_{n-m}(x, y_m)|}{|\partial_y f_{n-m}(x_m, y_m)|} < \frac{20}{9} \cdot \alpha^{1-\frac{4}{3}\eta_b}\sigma^{m-n}.$$

Since $G_m$ is connected and $G_m \subset I_m \times \mathbb{R}$, the inequality above implies that $G_m \subset R_m$, which completes the induction. As a result, from the skew-product structure of $F$ we know that $F^n : G_0 \to G_n$ is a diffeomorphism. Since $\det(\mathrm{D}F^n) = (Q_a^n)' \cdot \partial_y f_n$, and since the distortion of $Q_a^n$ is bounded by 9 on $I_0$, the conclusion that the distortion of $F^n$ on $G_0$ is bounded by 10 follows from (5.3) for $m = 0$. □

In proving that hyperbolic times has positive density for almost every $F$-orbit, we need the following lemma.

**Lemma 5.2.4** ([ABV00, Lemma 3.1]). *Given real numbers $A > c_2 > c_1$, denote $\theta_0 := \frac{c_2-c_1}{A-c_1} \in (0, 1)$. If a sequence of real numbers $a_1, \cdots, a_N$ satisfy that*

$$\sum_{j=1}^{N} a_j \ge c_2 N \qquad \textit{and} \qquad a_j \le A,\ 1 \le j \le N,$$

*then there exists an integer $l \geq \theta_0 N$ and a sequence of times $1 \leq n_1 < \cdots < n_l \leq N$, such that*

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n) \quad \textit{for} \quad 0 \leq n < n_i \quad \textit{and} \quad 1 \leq i \leq l.$$

**Lemma 5.2.5.** *Let*

$$\sigma = \sigma_b^{\frac{\eta_b}{6}} > 1 \quad \textit{and} \quad \theta = \frac{\eta_b \log \sigma_b}{3(12\log 2 - \eta_b \log \sigma_b)} \in (0,1).$$

*Then there exists $r > 0$, such that when $\alpha > 0$ is small, for Lebesgue almost every $(x,y) \in \mathbf{I}_{\beta_a} \times \Lambda_b$, when $N \in \mathbb{N}$ is large, the number of $(r,\sigma)$ hyperbolic times among $\{1,\cdots,N\}$ is no less than $\theta N$.*

*Proof.* Firstly, according to Proposition 5.1.1, for $\sigma$ in the lemma, for Lebesgue almost every $(x,y)$,

$$\liminf_{n\to\infty} \frac{1}{n} \log |\partial_y f_n(x,y)| \geq 3\log\sigma > 0.$$

Then by Lemma 5.2.4, for $a_n = \log|2f_{n-1}(x,y)|$, $A = \log 4$, $c_1 = \log\sigma$ and $c_2 = 2c_1$, when $N$ is large, the density of $1 \leq n \leq N$ for which the latter condition in (5.2) holds is no less than $3\theta$.

Secondly, according to Proposition 4.3.1, for $\varepsilon = \frac{\theta}{3}\log\sigma$, when $\alpha > 0$ is small, $\delta(\varepsilon)$ can be chosen as $\alpha^{\frac{\eta_b}{3}}$, then for Lebesgue almost every $(x,y)$,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \log |f_i(x,y)|_{\alpha^{1-\frac{5}{3}\eta_b}} \geq -\frac{\theta}{3}\log\sigma.$$

Here the below for $t \in \mathbb{R}$ and $\delta > 0$, we denote

$$|t|_\delta := \begin{cases} |t| & , \quad \text{if } |t| < \delta \\ 1 & , \quad \text{if } |t| \geq \delta \end{cases}.$$

Then by Lemma 5.2.4, for $a_n = \log|f_n(x,y)|_{\alpha^{1-\frac{5}{3}\eta_b}}$, $A = 0$, $c_1 = -\frac{1}{2}\log\sigma$ and $c_2 = -\frac{\theta}{2}\log\sigma$, when $N$ is large, the density of $1 \le n \le N$ for which the latter condition in (5.2) holds is no less than $1 - \theta$.

Combining the results in the two preceding paragraphs, we see that for Lebesgue almost every $(x, y)$, when $N$ is large, the density of $1 \le n \le N$ satisfying (5.2) is no less than $2\theta$. To complete the proof, it suffices to show that as $r \to 0^+$, for almost every $x \in \mathbf{I}_{\beta_a}$, when $N$ is large, the density of $r$-admissible pairs among $\{(x, n) : 1 \le n \le N\}$ can be arbitrarily close to 1.

Let $\mu_a$ be the unique a.c.i.p. of $Q_a$. Since the density of $\mu_a$ is $L^p$ integrable when $p < 2$, by Hölder's inequality, for every $\delta > 0$,

$$C_\delta := \int_{\mathbf{I}_{\beta_a}} \log \frac{1}{|x|_\delta} \, \mathrm{d}\mu_a(x) \in (0, +\infty) \quad \text{and} \quad \lim_{\delta \to 0^+} C_\delta = 0.$$

Then by Birkhoff's ergodic theorem, for a.e. $x \in \mathbf{I}_{\beta_a}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |Q_a^k(x)|_\delta = -C_\delta.$$

Therefore, by Lemma 5.2.4, for $a_n = \log |Q_a^{n-1}(x)|_\delta$, $A = 0$, $c_1 = -\frac{\log \lambda_a}{2}$ and $c_2 = -2C_\delta$, when $N$ is large, the density of $1 \le n \le N$ for which the following holds is no less than $\theta_\delta = 1 - \frac{4C_\delta}{\log \lambda_a}$:

$$\sum_{k=j}^{n-1} \log \frac{1}{|Q_a^k(x)|_\delta} \le \frac{1}{2} \log \lambda_a \cdot (n - j) \Longrightarrow |Q_a^j(x)| \ge \lambda_a^{-(n-j)/2} \wedge \delta, \quad 0 \le j \le n - 1. \quad (5.4)$$

Given $\delta > 0$, according to assertion (4) in Corollary 2.1.6, there exists $r > 0$, such that

$$|\operatorname{comp}(Q_a^j(x), n - j, 2r)| < \lambda_a^{-(n-j)}\delta \Longrightarrow 0 \notin \operatorname{comp}(Q_a^j(x), n - j, 2r), \quad 0 \le j \le n.$$

That is to say, once (5.4) holds for some $\delta > 0$, then there exists $r = r(\delta) > 0$, such that $(x, n)$ is an $r$-admissible pair. The conclusion follows as we choose $\delta$ so small that $\theta_\delta > 1 - \theta$ and determine the corresponding $r > 0$. □

The following lemma was proved in [ABV00, Lemma 5.6].

**Lemma 5.2.6.** *Let $M$ be a Riemannian manifold and let* Leb *denote the volume measure on $M$ induced by its metric. Let $f : M \to M$ be differentiable and let $G \subset M$ be an $f$-invariant measurable set with* $\mathrm{Leb}(G) > 0$*. Suppose that there are $\delta > 0$, $C > 1$ and* $\theta \in (0, 1)$*, such that for Lebesgue almost every $p \in G$, it satisfies the following property: when $n \in \mathbb{N}$ is large, there are $1 \le t_1 < \cdots < t_k \le n$ for some $k \ge \theta n$, such that for $i = 1, \cdots, k$, $f^{t_i}$ maps an open neighborhood of $p$ diffeomorphically onto the open ball centered at $f^{t_i}(p)$ of radius $\delta$ with distortion bound by $C$. Then there exists an open ball $\Delta$ of radius no less than $\frac{\delta}{4}$, such that* $\mathrm{Leb}(\Delta \setminus G) = 0$.

Then to complete the proof of Proposition 5.2.1, let us apply Lemma 5.2.6 to $F$ with $G$ being the basin of an a.c.i.p. of $F$.

*Proof of Proposition 5.2.1.* Let $G$ be the basin of an a.c.i.p. of $F$, which is automatically both $F$-invariant and of positive Lebesgue measure. Then Lemma 5.2.3 says that given $p \in G$, when $n \in \mathbb{N}$ is large, $1 < t_1 < \cdots < t_k < n$ in Lemma 5.2.6 can be chosen as $(r, \sigma)$-hyperbolic times of $p$; Lemma 5.2.5 says that with suitable choice of $r$, $\sigma$ and $\theta$, for almost every $p \in G$, $k \ge \theta n$ when $n$ is large. As a result, there exists a rectangle $R_0 = I_0 \times J_0$, such that $\mathrm{Leb}(R_0 \setminus G) = 0$ and $|J_0| > \alpha^{1-\frac{5}{4}\eta_b}$, provided that $\alpha > 0$ is small. Then by Lemma 2.3.2 with facts that $Q_a$ acts cyclically on $\check{I}_a^j$, $0 \le j < \hat{p}_a$, we know that there exists $n \in \mathbb{N}$, such that $F^n(R_0)$ contains a rectangle $R = \check{I}_a \times J$ with $|\check{I}_b^k \setminus J| \le \alpha^{0.9}$ for some $k \in \mathbb{N}_0$. Since $G$ is $F$-invariant and since $F$ preserves measure zero sets, the conclusion follows. □

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# VIANA MAPS DRIVEN BY BENEDICKS-CARLESON MAPS

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